Appendix A8

Hamiltonian dynamics

The symplectic structure of Hamilton's equations buys us much more than the incompressibility, or the phase space volume conservation alluded to in sect. 8.1.

A8.1 Stability of Hamiltonian flows

(M.J. Feigenbaum and P. Cvitanović)

The evolution equations for any p, q dependent quantity $Q = Q(q, p)$ are given by (19.28). In terms of the Poisson brackets, the time-evolution equation for $Q = Q(q, p)$ is given by (19.30). We now recast the symplectic condition (8.6) in a form convenient for using the symplectic constraints on M. Writing $x(t) = x' = [p', q']$ and the Jacobian matrix and its inverse

$$M = \begin{bmatrix} \frac{\partial q'}{\partial q} & \frac{\partial q'}{\partial p} \\ \frac{\partial p'}{\partial q} & \frac{\partial p'}{\partial p} \end{bmatrix}, \qquad M^{-1} = \begin{bmatrix} \frac{\partial q}{\partial q'} & \frac{\partial q}{\partial p'} \\ \frac{\partial p}{\partial q'} & \frac{\partial p}{\partial p'} \end{bmatrix}, \tag{A8.1}$$

we can spell out the symplectic invariance condition (8.6):

$$\begin{aligned} \frac{\partial q'_k}{\partial q_i}\frac{\partial p'_k}{\partial q_j} - \frac{\partial p'_k}{\partial q_i}\frac{\partial q'_k}{\partial q_j} &= 0 \\ \frac{\partial q'_k}{\partial p_i}\frac{\partial p'_k}{\partial p_j} - \frac{\partial p'_k}{\partial p_i}\frac{\partial q'_k}{\partial p_j} &= 0 \\ \frac{\partial q'_k}{\partial q_i}\frac{\partial p'_k}{\partial p_j} - \frac{\partial p'_k}{\partial q_i}\frac{\partial q'_k}{\partial p_j} &= \delta_{ij} \,. \end{aligned} \tag{A8.2}$$

From (8.20) we obtain

$$\frac{\partial q_i}{\partial q'_j} = \frac{\partial p'_j}{\partial p_i}, \quad \frac{\partial p_i}{\partial p'_j} = \frac{\partial q'_j}{\partial q_i}, \quad \frac{\partial q_i}{\partial p'_j} = -\frac{\partial q'_j}{\partial p_i}, \quad \frac{\partial p_i}{\partial q'_j} = -\frac{\partial p'_j}{\partial q_i} \,. \tag{A8.3}$$

Taken together, (A8.3) and (A8.2) imply that the flow conserves the $\{p, q\}$ Poisson brackets

$$\begin{aligned} \{q_i, q_j\} &= \frac{\partial q_i}{\partial p'_k}\frac{\partial q_j}{\partial q'_k} - \frac{\partial q_j}{\partial p'_k}\frac{\partial q_i}{\partial q'_k} = 0 \\ \{p_i, p_j\} &= 0\,, \qquad \{p_i, q_j\} = \delta_{ij}\,, \end{aligned} \tag{A8.4}$$

i.e., the transformations induced by a Hamiltonian flow are *canonical*, preserving the form of the equations of motion. The first two relations are symmetric under i, j interchange and yield $D(D-1)/2$ constraints each; the last relation yields D^2 constraints. Hence only $(2D)^2 - 2D(D-1)/2 - D^2 = d(2D+1)$ elements of M are linearly independent, as it behooves group elements of the symplectic group $Sp(2D)$.

We have now succeeded in making the full set of constraints explicit - as we shall see in appendix A19, this will enable us to implement dynamics in such a way that the symplectic invariance will be automatically preserved.

A8.2 Monodromy matrix for Hamiltonian flows

(G. Tanner)

It is not the Jacobian matrix J of the flow (4.5), but the *monodromy matrix* M, which enters the trace formula. This matrix gives the time dependence of a displacement perpendicular to the flow on the energy manifold. Indeed, we discover some trivial parts in the Jacobian matrix J. An initial displacement in the direction of the flow $x = \omega\nabla H(x)$ transfers according to $\delta x(t) = x_t(t)\delta t$ with δt time independent. The projection of any displacement on δx on $\nabla H(x)$ is constant, i.e., $\nabla H(x(t))\delta x(t) = \delta E$. We get the equations of motion for the monodromy matrix directly choosing a suitable local coordinate system on the orbit $x(t)$ in form of the (non singular) transformation $\mathbf{U}(x(t))$:

$$\tilde{J}(x(t)) = \mathbf{U}^{-1}(x(t))\, J(x(t))\, \mathbf{U}(x(0)) \tag{A8.5}$$

These lead to

$$\begin{aligned} \dot{\tilde{J}} &= \tilde{\mathbf{L}}\,\tilde{J} \\ \text{with} \qquad \tilde{\mathbf{L}} &= \mathbf{U}^{-1}(\mathbf{L}\mathbf{U} - \dot{\mathbf{U}}) \end{aligned} \tag{A8.6}$$

Note that the properties a) – c) are only fulfilled for $\tilde{J}$ and $\tilde{\mathbf{L}}$ if $\mathbf{U}$ itself is symplectic.

Choosing $x_E = \nabla H(t)/|\nabla H(t)|^2$ and x_t as local coordinates uncovers the two trivial eigenvalues 1 of the transformed matrix in (A8.5) at any time t. Setting

$\mathbf{U} = (x_t{}^\top, x_E{}^\top, x_1{}^\top, \ldots, x_{2d-2}{}^\top)$ gives

$$\tilde{J} = \begin{pmatrix} 1 & * & * & \ldots & * \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & * & & & \\ \vdots & \vdots & & \mathbf{M} & \\ 0 & * & & & \end{pmatrix}; \qquad \tilde{\mathbf{L}} = \begin{pmatrix} 0 & * & * & \ldots & * \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & * & & & \\ \vdots & \vdots & & \mathbf{l} & \\ 0 & * & & & \end{pmatrix}, \tag{A8.7}$$

The matrix $\mathbf{M}$ is now the monodromy matrix and the equation of motion are given by

$$\dot{\mathbf{M}} = \mathbf{l}\,\mathbf{M}. \tag{A8.8}$$

The vectors $x_1, \ldots, x_{2d-2}$ must span the space perpendicular to the flow on the energy manifold.

For a system with two degrees of freedom, the matrix $\mathbf{U}(\mathbf{t})$ can be written down explicitly, i.e.,

$$\mathbf{U}(t) = (x_t, x_1, x_E, x_2) = \begin{pmatrix} \dot{x} & -\dot{y} & -\dot{u}/q^2 & -\dot{v}/q^2 \\ \dot{y} & \dot{x} & -\dot{v}/q^2 & \dot{u}/q^2 \\ \dot{u} & \dot{v} & \dot{x}/q^2 & -\dot{y}/q^2 \\ \dot{v} & -\dot{u} & \dot{y}/q^2 & \dot{x}/q^2 \end{pmatrix} \tag{A8.9}$$

with $x^\top = (x, y; u, v)$ and $q = |\nabla H| = |\dot{x}|$. The matrix $\mathbf{U}$ is non singular and symplectic at every phase space point x, except the equilibrium points $\dot{x} = 0$. The matrix elements for $\mathbf{l}$ are given (A8.11). One distinguishes 4 classes of eigenvalues of $\mathbf{M}$.

- *stable* or *elliptic*, if $\Lambda = e^{\pm i\pi\nu}$ and $\nu \in]0, 1[$.
- *marginal*, if $\Lambda = \pm 1$.
- *hyperbolic*, *inverse hyperbolic*, if $\Lambda = e^{\pm\lambda}$, $\Lambda = -e^{\pm\lambda}$.
- *loxodromic*, if $\Lambda = e^{\pm\mu\pm i\omega}$ with μ and ω real. This is the most general case, possible only in systems with 3 or more degree of freedoms.

For 2 degrees of freedom, i.e., $\mathbf{M}$ is a [2×2] matrix, the eigenvalues are determined by

$$\lambda = \frac{\mathrm{tr}\,(\mathbf{M}) \pm \sqrt{\mathrm{tr}\,(\mathbf{M})^2 - 4}}{2}, \tag{A8.10}$$

i.e., $\mathrm{tr}\,(\mathbf{M}) = 2$ separates stable and unstable behavior.

The **l** matrix elements for the local transformation (A8.9) are

$$
\begin{aligned}
\tilde{\mathbf{l}}_{11} &= \frac{1}{q}[(h_x^2 - h_y^2 - h_u^2 + h_v^2)(h_{xu} - h_{yv}) + 2(h_x h_y - h_u h_v)(h_{xv} + h_{yu}) \\
&\quad -(h_x h_u + h_y h_v)(h_{xx} + h_{yy} - h_{uu} - h_{vv})] \\
\tilde{\mathbf{l}}_{12} &= \frac{1}{q^2}[(h_x^2 + h_v^2)(h_{yy} + h_{uu}) + (h_y^2 + h_u^2)(h_{xx} + h_{vv}) \\
&\quad -2(h_x h_u + h_y h_v)(h_{xu} + h_{yv}) - 2(h_x h_y - h_u h_v)(h_{xy} - h_{uv})] \\
\tilde{\mathbf{l}}_{21} &= -(h_x^2 + h_y^2)(h_{uu} + h_{vv}) - (h_u^2 + h_v^2)(h_{xx} + h_{yy}) \\
&\quad +2(h_x h_u - h_y h_v)(h_{xu} - h_{yv}) + 2(h_x h_v + h_y h_u)(h_{xv} + h_{yu}) \\
\tilde{\mathbf{l}}_{22} &= -\tilde{\mathbf{l}}_{11},
\end{aligned}
\qquad (A8.11)
$$

with h_i, h_{ij} is the derivative of the Hamiltonian H with respect to the phase space coordinates and $q = |\nabla H|^2$.