Chapter 10

Flips, slides and turns

A detour of a thousand pages starts with a single misstep.
—Chairman Miaw

DYNAMICAL SYSTEMS often come equipped with symmetries, such as the reflection and rotation symmetries of various potentials.

This chapter assumes familiarity with basic group theory, as discussed in appendix A7.1. We find the abstract notions easier to digest by working out the examples; links to these examples are interspersed throughout the chapter. Working through these examples is essential and will facilitate your understanding of various definitions. The erudite reader might prefer to skip the lengthy group-theoretic overture and go directly to $C_2 = D_1$ example 11.3, example 11.5, and $C_{3v} = D_3$ example 10.1, backtrack as needed.

10.1 Discrete symmetries

We show that a symmetry equates multiplets of equivalent orbits, or 'stratifies' the state space into equivalence classes, each class a 'group orbit'. We start by defining a finite (discrete) group, its state space representations, and what we mean by a *symmetry* (*invariance* or *equivariance*) of a dynamical system. As is always the problem with 'gruppenpest' (read appendix A1.6) way too many abstract notions have to be defined before an intelligent conversation can take place. Perhaps best to skim through this section on the first reading, then return to it later as needed.

Definition: A **group** consists of a set of elements

$$G = \{e, g_2, \ldots, g_n, \ldots\} \tag{10.1}$$

and a group multiplication rule $g_j \circ g_i$ (often abbreviated as $g_j g_i$), satisfying

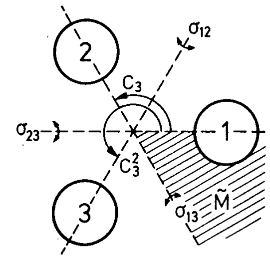


Figure 10.1: The symmetries of three disks on an equilateral triangle. A fundamental domain is indicated by the shaded wedge. Work through example 10.1.

1. Closure: If $g_i, g_j \in G$, then $g_j \circ g_i \in G$
2. Associativity: $g_k \circ (g_j \circ g_i) = (g_k \circ g_j) \circ g_i$
3. Identity e: $g \circ e = e \circ g = g$ for all $g \in G$
4. Inverse g^{-1}: For every $g \in G$, there exists a unique element $h = g^{-1} \in G$ such that $h \circ g = g \circ h = e$.

If the group is finite, the number of elements, $|G| = n$, is called the *order* of the group.

The theory of finite groups is developed on two levels. There is a beautiful theory of *groups* as abstract entities which yields the classification of their structures and their irreducible, orthogonal representations in terms of characters. Then there is the considerably messier matter of *group representations*, in our case the ways in which a given symmetry group acts on and stratifies the particular state space of a problem at hand, the most familiar being the ways in which symmetries reduce and block-diagonalize quantum-mechanical problems. What helps us here is that the symmetries 'commute' with dynamics, i.e., we can first reduce a given state space to its irreducible components, using the symmetry alone, and then study the action of dynamics on these subspaces. As our intuition is based on physical manifestations of group actions, in this brief review we shall freely switch gears between the abstract and the representation levels whenever pedagogically convenient.

For example, do work through example 10.1. Once you understand how this works out for the symmetries of an equilateral triangle, or, equivalently, for the three disk billiard of figure 10.1, you know almost everything you need to know about the general, non-abelian finite groups. example A7.1
example A7.2

example 10.1
p. 171

example 10.3
p. 171

Definition: Coordinate transformations. Consider a map $x' = f(x)$, $x, x' \in \mathcal{M}$. An *active* coordinate transformation Mx corresponds to a non-singular $[d\times$

$d]$ matrix M that maps the initial vector $x \in \mathcal{M}$ onto another vector $Mx \in \mathcal{M}$. The corresponding *passive* coordinate transformation $x' \to M^{-1}x'$ changes the coordinate system with respect to which the final vector $x' \in \mathcal{M}$ is measured. Together, a passive and active coordinate transformations yield the map in the transformed coordinates:

$$\hat{f}(x) = M^{-1} f(Mx)\,. \tag{10.2}$$

(For general nonlinear coordinate transformations, see Appendix A2.)

Definition: Matrix group. The set of $[d\times d]$-dimensional real non-singular matrices $A, B, C, \cdots \in GL(d)$ acting in a d-dimensional vector space $V \in \mathbb{R}^d$ forms the general linear group $GL(d)$ under matrix multiplication. The product of matrices A and B gives the matrix C, $Cx = B(Ax) = (BA)x \in V$, for all $x \in V$. The unit matrix $\mathbf{1}$ is the identity element which leaves all vectors in V unchanged. Every matrix in the group has a unique inverse.

Definition: Matrix representation. Linear action of a group element g on states $x \in \mathcal{M}$ is given by a finite non-singular $[d\times d]$ matrix $D(g)$, the *matrix representation* of element $g \in G$. For brevity we shall often denote by 'g' both the abstract group element and its matrix representation, $D(g)x \to gx$.

However, when dealing simultaneously with several representations of the same group action, the notation $D^{(\mu)}(g)$ is preferable, where μ is a representation label, (see appendix A7.1). A linear or matrix representation $\mathsf{D}(G)$ of the abstract group G acting on a *representation space* V is a group of matrices $\mathsf{D}(G)$ such that

1. Any $g \in G$ is mapped to a matrix $D(g) \in \mathsf{D}(G)$.
2. The group product $g_2 \circ g_1$ is mapped onto the matrix product $D(g_2 \circ g_1) = D(g_2)D(g_1)$.
3. The associativity follows from the associativity of matrix multiplication, $D(g_3 \circ (g_2 \circ g_1)) = D(g_3)(D(g_2)D(g_1)) = (D(g_3)(D(g_2))D(g_1)$.
4. The identity element $e \in G$ is mapped onto the unit matrix $D(e) = 1$ and the inverse element $g^{-1} \in G$ is mapped onto the inverse matrix $D(g^{-1}) = D(g)^{-1}$.

   Some simple 3D representations of the group order 2 are given in example 10.4.

example 10.4
p. 171

If the coordinate transformation g belongs to a linear non-singular representation of a discrete finite group G, for any element $g \in G$ there exists a number

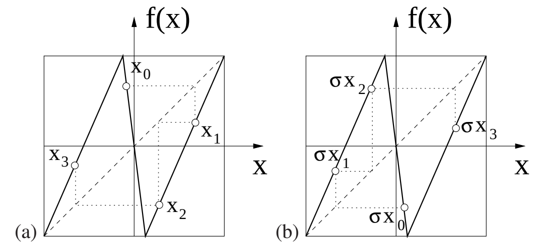


Figure 10.2: The bimodal Ulam sawtooth map with the D_1 symmetry $f(-x) = -f(x)$. If the trajectory (a) $x_0 \to x_1 \to x_2 \to \cdots$ is a solution, so is its reflection (b) $\sigma x_0 \to \sigma x_1 \to \sigma x_2 \to \cdots$. (work through example 10.5; continued in figure 11.1).

$m \le |G|$ such that

$$g^m \equiv \underbrace{g \circ g \circ \cdots \circ g}_{m \text{ times}} = e \quad \to \quad |\det D(g)| = 1\,. \tag{10.3}$$

As the modulus of its determinant is unity, $\det g$ is an mth root of 1. This is the reason why all finite groups have unitary representations.

Definition: Symmetry of a dynamical system. A group G is a *symmetry* of the dynamics if for every solution $f(x) \in \mathcal{M}$ and $g \in G$, $gf(x)$ is also a solution.

Another way to state this: A dynamical system $(\mathcal{M}, f)$ is *invariant* (or *G-equivariant*) under a symmetry group G if the time evolution $f : \mathcal{M} \to \mathcal{M}$ (a discrete time map f, or the continuous flow f^t map from the d-dimensional manifold $\mathcal{M}$ into itself) commutes with all actions of G,

$$f(gx) = gf(x)\,. \tag{10.4}$$

In the language of physicists: The 'law of motion' is invariant, i.e., retains its form in any symmetry-group related coordinate frame (10.2),

$$f(x) = g^{-1} f(gx)\,, \tag{10.5}$$

for $x \in \mathcal{M}$ and *any* finite non-singular $[d\times d]$ matrix representation g of element $g \in G$. As these are true *any* state x, one can state this more compactly as $f \circ g = g \circ f$, or $f = g^{-1} \circ f \circ g$.

Why 'equivariant?' A scalar function $h(x)$ is said to be *G-invariant* if $h(x) = h(gx)$ for all $g \in G$. The group actions map the solution $f : \mathcal{M} \to \mathcal{M}$ into different (but equivalent) solutions $gf(x)$, hence the invariance condition $f(x) = g^{-1}f(gx)$ appropriate to vectors (and, more generally, tensors). The full set of such solutions is *G-invariant*, but the flow that generates them is said to be *G*-equivariant. It is obvious from the context, but for verbal emphasis applied mathematicians like to distinguish the two cases by *in/equi*-variant. The distinction is helpful in distinguishing the dynamics written in the original, equivariant coordinates from the dynamics rewritten in terms of *invariant* coordinates, see sects. 11.4 and 13.2.

exercise 11.2

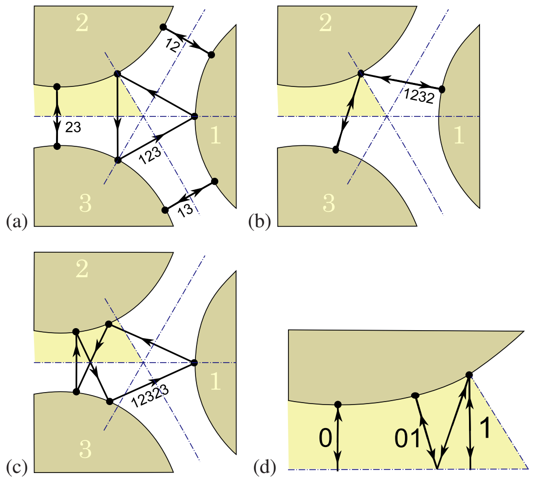


Figure 10.3: The 3-disk pinball cycles: (a) $\overline{12}$, $\overline{13}$, $\overline{23}$, $\overline{123}$; the clockwise $\overline{132}$ not drawn. (b) Cycle $\overline{1232}$; the symmetry related $\overline{1213}$ and $\overline{1323}$ not drawn. (c) Cycle $\overline{12323}$; cycles $\overline{12123}$, $\overline{12132}$, $\overline{12313}$, $\overline{13131}$ and $\overline{13232}$ not drawn. (d) The fundamental domain, i.e., the light-shaded 1/6th wedge in (a), consisting of a section of a disk, two segments of symmetry axes acting as straight mirror walls, and the escape gap to the left. The above 14 full-space cycles restricted to the fundamental domain and recoded in binary reduce to the two fixed points $\overline{0}$, $\overline{1}$, 2-cycle $\overline{10}$, and 5-cycle $\overline{00111}$ (not drawn). See figure 11.6 for the $\overline{001}$ cycle. Work through example 10.2.

example 10.5 p. 172

example 10.6 p. 172

example 10.2 p. 171

example 12.9 p. 207

10.2 Subgroups, cosets, classes

Normal is just a setting on a washing machine.
—Borgette, Borgo's daughter

Inspection of figure 10.3 indicates that various 3-disk orbits are the same up to a symmetry transformation. Here we set up some group-theoretic notions needed to describe such relations. The reader might prefer to skip to sect. 11.1, backtrack as needed.

Definition: Subgroup. A set of group elements $H = \{e, b_2, b_3, \ldots, b_h\} \subseteq G$ closed under group multiplication forms a subgroup.

Definition: Coset. Let $H = \{e, b_2, b_3, \ldots, b_h\} \subseteq G$ be a subgroup of order $h = |H|$. The set of h elements $\{c, cb_2, cb_3, \ldots, cb_h\}$, $c \in G$ but not in H, is called left *coset* cH. For a given subgroup H the group elements are partitioned into H and $m-1$ cosets, where $m = |G|/|H|$. The cosets *cannot be* subgroups, since they do

not include the identity element. A nontrival subgroup can exist only if $|G|$, the order of the group, is divisible by $|H|$, the order of the subgroup, i.e., only if $|G|$ is not a prime number.

example 10.7 p. 172

Next we need a notion that will, for example, identify the three 3-disk 2-cycles in figure 10.3 as belonging to the same class.

Definition: Class. An element $b \in G$ is *conjugate* to a if $b = c\,a\,c^{-1}$ where c is some other group element. If b and c are both conjugate to a, they are conjugate to each other. Application of all conjugations separates the set of group elements into mutually not-conjugate subsets called *classes*, *types* or *conjugacy classes*. The identity e is always in the class $\{e\}$ of its own. This is the only class which is a subgroup, all other classes lack the identity element.

exercise 10.3
exercise 10.5

example 10.8 p. 172

The geometrical significance of classes is clear from (10.5); it is the way coordinate transformations act on mappings. The action, such as a reflection or rotation, of an element is equivalent to redefining the coordinate frame.

Definition: Conjugate symmetry subgroups. The splitting of a group G into a symmetry group G_p of orbit $\mathcal{M}_p$ and $m_p - 1$ cosets cG_p relates the orbit $\mathcal{M}_p$ to m_p-1 other distinct orbits $c\mathcal{M}_p$. All of them have equivalent symmetry subgroups, or, more precisely, the points on the same group orbit have *conjugate symmetry subgroups* (or *conjugate stabilizers*):

exercise 10.4

$$G_{c\,p} = c\,G_p\,c^{-1}\,, \tag{10.6}$$

i.e., if G_p is the symmetry of orbit $\mathcal{M}_p$, elements of the coset space $c \in G/G_p$ generate the $m_p - 1$ distinct copies of $\mathcal{M}_p$.

Definition: Invariant subgroup. A subgroup $H \subseteq G$ is an *invariant* subgroup or *normal divisor* if it consists of complete classes. Class is complete if no conjugation takes an element of the class out of H.

Think of action of H within each coset as identifying its $|H|$ elements as equivalent. This leads to the notion of the *factor group* or *quotient group* G/H of G, with respect to the invariant subgroup H. H thus divides G into H and $m-1$ cosets, each of order $|H|$. The order of G/H is $m = |G|/|H|$, and its multiplication table can be worked out from the G multiplication table class by class, with the subgroup H playing the role of identity. G/H is *homeomorphic* to G, with $|H|$ elements in a class of G represented by a single element in G/H.

10.3 Orbits, quotient space

section 2.1

Definition: Orbit. The subset $\mathcal{M}_{x_0} \subset \mathcal{M}$ traversed by the infinite-time trajectory of a given point x_0 is called the *orbit* (or *time orbit*, or *solution*) $x(t) = f^t(x_0)$. An orbit is a *dynamically invariant* notion: it refers to the set of all states that can be reached in time from x_0, thus as a set it is invariant under time evolution. The full state space $\mathcal{M}$ is a union of such orbits. We label a generic orbit $\mathcal{M}_{x_0}$ by any point belonging to it, $x_0 = x(0)$ for example.

A generic orbit might be ergodic, unstable and essentially uncontrollable. The ChaosBook strategy is to populate the state space by a hierarchy of orbits which are *compact invariant sets* (equilibria, periodic orbits, invariant tori, ...), each computable in a finite time. They are a set of zero Lebesgue measure, but dense on the non–wandering set, and are to a generic orbit what fractions are to normal numbers on the unit interval. We label orbits confined to compact invariant sets by whatever alphabet we find convenient in a given context: point $EQ = x_{EQ} = \mathcal{M}_{EQ}$ for an equilibrium, 1-dimensional loop $p = \mathcal{M}_p$ for a prime periodic orbit p, etc. (note also discussion on page 202, and the distinction between trajectory and orbit made in sect. 2.1; a trajectory is a finite-time segment of an orbit).

Definition: Group orbit or the *G-orbit* of the point $x \in \mathcal{M}$ is the set

$$\mathcal{M}_x = \{g\,x \mid g \in G\} \tag{10.7}$$

of all state space points into which x is mapped under the action of G. If G is a symmetry, intrinsic properties of an equilibrium (such as stability eigenvalues) or a cycle p (period, Floquet multipliers) evaluated anywhere along its G-orbit are the same.

A symmetry thus reduces the number of inequivalent solutions $\mathcal{M}_p$. So we also need to describe the symmetry of a *solution*, as opposed to (10.5), the symmetry of the *system*.

Definition: Reduced state space. The action of group G partitions the state space $\mathcal{M}$ into a union of group orbits. This set of group orbits, denoted $\mathcal{M}/G$, has many names: *reduced state space*, *quotient space* or any of the names listed on page 214.

Definition: Fundamental domain. The images of a single point x under all actions of a discrete group G form a G-orbit $\mathcal{M}_x$. A fundamental domain $\hat{\mathcal{M}} = \mathcal{M}/G$ is a subset of the state space $\mathcal{M}$ which contains exactly one point from each G-orbit. It is an explicit state space realization of the abstract notion of the reduced state space $\mathcal{M}/G$ in the case that G is a discrete group.

A fundamental domain can be defined in different ways, here exemplified by figure 10.1, figure 10.3, figure 11.5, figure 11.6, figure 11.3 (a) and figure 24.2.

Ideally it is a connected subset with restrictions on its boundary that ensure the no points are double-counted. The set of images of a fundamental domain under the group action then tiles the entire state space.

Reduction of the dynamical state space is discussed in sect. 11.3 for discrete symmetries, and in sect. 13.2 for continuous symmetries.

Definition: Fixed-point subspace. $\mathcal{M}_H$ is the set of all state space points left *H-fixed*, *point-wise* invariant under subgroup or 'centralizer' $H \subseteq G$ action

$$\mathcal{M}_H = \mathrm{Fix}\,(H) = \{x \in \mathcal{M} \mid h\,x = x \text{ for all } h \in H\}\,. \tag{10.8}$$

Points in state space subspace $\mathcal{M}_G$ which are fixed points of the full group action are called *invariant points*,

$$\mathcal{M}_G = \mathrm{Fix}\,(G) = \{x \in \mathcal{M} \mid g\,x = x \text{ for all } g \in G\}\,. \tag{10.9}$$

Definition: Flow invariant subspace. A typical point in fixed-point subspace $\mathcal{M}_H$ moves with time, but, due to equivariance (10.4), its trajectory $x(t) = f^t(x)$ remains within $f(\mathcal{M}_H) \subseteq \mathcal{M}_H$ for all times,

$$h f^t(x) = f^t(hx) = f^t(x)\,, \quad h \in H\,, \tag{10.10}$$

i.e., it belongs to a *flow invariant subspace*. This suggests a systematic approach to seeking compact invariant solutions. The larger the symmetry subgroup, the smaller $\mathcal{M}_H$, easing the numerical searches, so start with the largest subgroups H first.

We can often decompose the state space into smaller subspaces, with group acting within each 'chunk' separately:

Definition: Invariant subspace. $\mathcal{M}_\alpha \subset \mathcal{M}$ is an *invariant* subspace if

$$\{\mathcal{M}_\alpha \mid gx \in \mathcal{M}_\alpha \text{ for all } g \in G \text{ and } x \in \mathcal{M}_\alpha\}\,. \tag{10.11}$$

$\{0\}$ and $\mathcal{M}$ are always invariant subspaces. So is any $\mathrm{Fix}\,(H)$ which is point-wise invariant under action of G.

Definition: Irreducible subspace. A space $\mathcal{M}_\alpha$ whose only invariant subspaces under the action of G are $\{0\}$ and $\mathcal{M}_\alpha$ is called *irreducible*.

Definition: Reducibility. If state space $\mathcal{M}$ on which G acts can be written as a direct sum of irreducible subspaces, then the representation of G on state space $\mathcal{M}$ is completely reducible.

This being group theory, definitions could go on forever. But we stop here, hopefully having defined everything that we need at the moment, and we pile on a few more definitions in sect. 11.1 and chapter 12. There is also appendix A7.1, if you thirst for more.

Résumé

A group G is a *symmetry* of the dynamical system $(\mathcal{M}, f)$ if its 'law of motion' retains its form under all symmetry-group actions, $f(x) = g^{-1} f(gx)$. A mapping f is said to be *invariant* if $gf = f$, where g is any element of G. If the mapping and the group actions commute, $gf = fg$, f is said to be *equivariant*. The governing dynamical equations are equivariant with respect to the symmetry group G.

Commentary

Remark 10.1 Literature. We found Tinkham [26.6] the most enjoyable as a no-nonsense, the user friendliest introduction to the basic concepts. Slightly longer, but perhaps student-friendlier is *Part I Basic Mathematics* of Dresselhaus *et al.* [25.3]. Byron and Fuller [10.2], the last chapter of volume two, offers an introduction even more compact than Tinkham's. For a summary of the theory of discrete groups see, for example, ref. [10.3]. Chapter 3 of Rebecca Hoyle [12.2] is a very student-friendly overview of the group theory a nonlinear dynamicist might need, with exception of the quotienting, reduction of dynamics to a fundamental domain, which is not discussed at all. For that, Fundamental domain wiki is very clear. We also found Quotient group wiki helpful. Curiously, we have not read any of the group theory books that Hoyle recommends as background reading, which just confirms that there are way too many group theory books out there. For example, one that you will not find useful at all is ref. [26.14]. The reason is presumably that in the 20th century physics (which motivated much of the work on the modern group theory) the focus is on the linear representations used in quantum appendix A1.6 mechanics, crystallography and quantum field theory. We shall need these techniques in Chapter 25, where we reduce the linear action of evolution operators to irreducible subspaces. However, here we are looking at nonlinear dynamics, and the emphasis is on the symmetries of orbits, their reduced state space sisters, and the isotypic decomposition of their linear stability matrices.

In ChaosBook we focus on chaotic dynamics, and skirt the theory of bifurcations, the landscape between the boredom of regular motions and the thrills of chaos. Chapter 4 of Rebecca Hoyle [12.2] is a student-friendly introduction to the treatment of bifurcations in presence of symmetries, worked out in full detail and generality in monographs by Golubitsky, Stewart and Schaeffer [12.19], Golubitsky and Stewart [12.4] and Chossat and Lauterbach [13.22]. Term 'stabilizer' is used, for example, by Broer *et al.* [10.9] to refer to a periodic orbit with Z_2 symmetry; they say that the relative or pre-periodic segment is in this case called a 'short periodic orbit.' In Efstathiou [10.10] a subgroup of 'short periodic orbit' symmetries is referred to as a 'nontrivial isotropy group or stabilizer.' Chap. 8 of Govaerts [10.11] offers a review of numerical methods that employ equivariance with respect to compact, and mostly discrete groups. (continued in remark 12.1)

10.4 Examples

Example 10.1 ***$C_{3v} = D_3$ symmetry of the 3-disk game of pinball:*** *If the three unit-radius disks in figure 10.1 are equidistantly spaced, our game of pinball has a sixfold symmetry. The symmetry group of relabeling the 3 disks is the permutation group S_3; however, it is more instructive to think of this group geometrically, as C_{3v}, also known as the dihedral group*

$$D_3 = \{e, \sigma_{12}, \sigma_{13}, \sigma_{23}, C^{1/3}, C^{2/3}\}, \tag{10.12}$$

the group of order $|G| = 6$ consisting of the identity element e, three reflections across symmetry axes $\{\sigma_{12}, \sigma_{23}, \sigma_{13}\}$, and two rotations by $2\pi/3$ and $4\pi/3$ denoted $\{C^{1/3}, C^{2/3}\}$. (continued in example 10.2) click to return: p. ??

Example 10.2 ***3-disk game of pinball - symmetry-related orbits:*** *(continued from example 10.1) Applying an element (identity, rotation by $\pm 2\pi/3$, or one of the three possible reflections) of this symmetry group to a trajectory yields another trajectory. For instance, σ_{23}, the flip across the symmetry axis going through disk 1 interchanges the symbols 2 and 3; it maps the cycle $\overline{12123}$ into $\overline{13132}$, figure 10.3 (c). Cycles $\overline{12}$, $\overline{23}$, and $\overline{13}$ in figure 10.3 (a) are related to each other by rotation by $\pm 2\pi/3$, or, equivalently, by a relabeling of the disks. (continued in example 10.7)* click to return: p. ??

Example 10.3 ***Discrete groups of order 2 on $\mathbb{R}^3$.*** *Three types of discrete group of order 2 can arise by linear action on our 3-dimensional Euclidean space $\mathbb{R}^3$:*

$$\begin{aligned} \text{reflections: } \sigma(x,y,z) &= (x,y,-z) \\ \text{rotations: } C^{1/2}(x,y,z) &= (-x,-y,z) \\ \text{inversions: } P(x,y,z) &= (-x,-y,-z). \end{aligned} \tag{10.13}$$

σ is a reflection *(or an inversion) through the $[x,y]$ plane. $C^{1/2}$ is $[x,y]$-plane, constant z* rotation *by π about the z-axis (or an inversion thorough the z-axis). P is an* inversion *(or parity operation) through the point $(0,0,0)$. Singly, each operation generates a group of order 2: $D_1 = \{e,\sigma\}$, $C_2 = \{e, C^{1/2}\}$, and $D_1 = \{e, P\}$. Together, they form the dihedral group $D_2 = \{e, \sigma, C^{1/2}, P\}$ of order 4. (continued in example 10.4)* click to return: p. ??

Example 10.4 ***Discrete operations on $\mathbb{R}^3$.*** *(continued from example 10.3) The matrix representation of reflections, rotations and inversions defined by (10.13) is*

$$D(\sigma) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad D(C^{1/2}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad D(P) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{10.14}$$

with $\det D(C^{1/2}) = 1$, $\det D(\sigma) = \det D(P) = -1$; that is why we refer to $C^{1/2}$ as a rotation, and σ, P as inversions. As $g^2 = e$ in all three cases, these are groups of order 2. (continued in example 10.6) click to return: p. ??

Example 10.5 ***A reflection symmetric $1d$ map.*** *Consider a $1d$ map f with reflection symmetry $f(-x) = -f(x)$, such as the bimodal 'sawtooth' map of figure 10.2, piecewise-linear on the state space $\mathcal{M} = [-1, 1]$, a compact 1-dimensional line interval, split into three regions $\mathcal{M} = \mathcal{M}_L \cup \mathcal{M}_C \cup \mathcal{M}_R$. Denote the reflection operation by $\sigma x = -x$. The 2-element group $G = \{e, \sigma\}$ goes by many names, such as Z_2 or C_2. Here we shall refer to it as D_1, dihedral group generated by a single reflection. The G-equivariance of the map implies that if $\{x_n\}$ is a trajectory, than also $\{\sigma x_n\}$ is a symmetry-equivalent trajectory because $\sigma x_{n+1} = \sigma f(x_n) = f(\sigma x_n)$ (continued in example 11.3)* click to return: p. ??

Example 10.6 ***Equivariance of the Lorenz flow.*** *(continued from example 10.4) The velocity field in Lorenz equations (2.22)*

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \sigma(y-x) \\ \rho x - y - xz \\ xy - bz \end{bmatrix}$$

is equivariant under the action of cyclic group $C_2 = \{e, C^{1/2}\}$ acting on $\mathbb{R}^3$ by a π rotation about the z axis,

$$C^{1/2}(x,y,z) = (-x,-y,z). \tag{10.15}$$

(continued in example 11.5) click to return: p. ??

Example 10.7 ***Subgroups, cosets of D_3:*** *(continued from example 10.2) The 3-disks symmetry group, the D_3 dihedral group (10.12) has six subgroups*

$$\{e\}, \quad \{e,\sigma_{12}\}, \{e,\sigma_{13}\}, \{e,\sigma_{23}\}, \quad \{e, C^{1/3}, C^{2/3}\}, \quad D_3. \tag{10.16}$$

The left cosets of subgroup $D_1 = \{e, \sigma_{12}\}$ are $\{\sigma_{13}, C^{1/3}\}$, $\{\sigma_{23}, C^{2/3}\}$. The coset of subgroup $C_3 = \{e, C^{1/3}, C^{2/3}\}$ is $\{\sigma_{12}, \sigma_{13}, \sigma_{23}\}$. The significance of the coset is that if a solution has a symmetry H, for example the symmetry of a 3-cycle $\overline{123}$ is C_3, then all elements in a coset act on it the same way, for example $\{\sigma_{12}, \sigma_{13}, \sigma_{23}\}\overline{123} = \overline{132}$.

The nontrivial subgroups of D_3 are $D_1 = \{e, \sigma\}$, consisting of the identity and any one of the reflections, of order 2, and $C_3 = \{e, C^{1/3}, C^{2/3}\}$, of order 3, so possible cycle multiplicities are $|G|/|G_p| = 1$, 2, 3 or 6. Only the fixed point at the origin has full symmetry $G_p = G$. Such equilibria exist for smooth potentials, but not for the 3-disk billiard. Examples of other multiplicities are given in figure 10.3 and figure 11.4. (continued in example 10.8) click to return: p. ??

Example 10.8 ***D_3 symmetry - classes:*** *(continued from example 10.7) The three classes of the 3-disk symmetry group $D_3 = \{e, C^{1/3}, C^{2/3}, \sigma, \sigma C^{1/3}, \sigma C^{2/3}\}$, are the identity, any one of the reflections, and the two rotations,*

$$\{e\}, \quad \begin{Bmatrix} \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{Bmatrix}, \quad \begin{Bmatrix} C^{1/3} \\ C^{2/3} \end{Bmatrix}. \tag{10.17}$$

In other words, the group actions either flip or rotate. (continued in example 11.4) click to return: p. ??

Exercises

10.1. **Polynomials invariant under discrete operations on $\mathbb{R}^3$.** Prove that the $\{e, \sigma\}$, $\{e, C^{1/2}\}$, $\{e, P\}$ and $\{e, \sigma, C^{1/2}, P\}$-invariant polynomial basis and syzygies are those listed in example 11.9.

10.2. **$G_x \subset G$.** The maximal set of group actions which maps a state space point x into itself,

$$G_x = \{g \in G : gx = x\}, \tag{10.18}$$

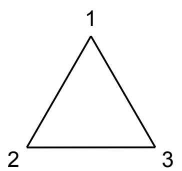
is called the *isotropy group* (or *stability subgroup* or *little group*) of x. Prove that the set G_x as defined in (10.18) is a subgroup of G.

10.3. **Transitivity of conjugation.** Assume that g_1, g_2, $g_3 \in G$ and both g_1 and g_2 are conjugate to g_3. Prove that g_1 is conjugate to g_2.

10.4. **Isotropy subgroup of gx.** Prove that for $g \in G$, x and gx have conjugate isotropy subgroups:

$$G_{gx} = g\, G_x\, g^{-1}$$

10.5. **D_3: symmetries of an equilateral triangle.** Consider group $D_3 \cong C_{3v}$, the symmetry group of an equilateral triangle:



(a) List the group elements and the corresponding geometric operations

(b) Find the subgroups of the group D_3.

(c) Find the classes of D_3 and the number of elements in them, guided by the geometric interpretation of group elements. Verify your answer using the definition of a class.

(d) List the conjugacy classes of subgroups of D_3.

(continued as exercise 12.2 and exercise 25.3)

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