herding cats

a chaotic field theory

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ChaosBook.org/overheads/spatiotemporal/kittens/ notes
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I have said it thrice:
What I tell you three times is true.

— Lewis Carrol, *The Hunting of the Snark*

1. coin toss
2. kicked rotor
3. **spatiotemporal cat**
4. bye bye, dynamics

spatiotemporally infinite ‘spatiotemporal cat’

herding cats in d spacetime dimensions

start with

a cat at each lattice site

talk to neighbors

spacetime d-dimensional spatiotemporal cat

- Hamiltonian formulation is awkward, fuggedaboutit!
- Lagrangian formulation is elegant

spatiotemporal cat

consider a 1 spatial dimension lattice, with field ϕ_{nt}
(the angle of a kicked rotor "particle" at instant t, at site n)

require

- each site couples to its nearest neighbors $\phi_{n\pm1,t}$
- invariance under spatial translations
- invariance under spatial reflections
- invariance under the space-time exchange

Gutkin & Osipov[1] obtain

2-dimensional coupled cat map lattice

$$\phi_{n,t+1} + \phi_{n,t-1} - 2s\,\phi_{nt} + \phi_{n+1,t} + \phi_{n-1,t} = -m_{nt}$$

[1] B. Gutkin and V. Osipov, Nonlinearity **29**, 325–356 (2016).

spatiotemporal cat : a strong coupling field theory

symmetries : translations ∘ time-reversal ∘ spatial reflections

the key assumption

- invariance under the space-time exchange

not a traditional
spatially weakly coupled lattice model[2]

- spatiotemporal cat is a Euclidean field theory

[2] L. A. Bunimovich and Y. G. Sinai, Nonlinearity **1**, 491 (1988).

herding cats : a discrete Euclidean space-time field theory

write the spatial-temporal differences as discrete derivatives

Laplacian in $d = 2$ dimensions

$$\Box\,\phi_{nt} \;=\; \phi_{n,t+1} + \phi_{n,t-1} - 4\,\phi_{nt} + \phi_{n+1,t} + \phi_{n-1,t}$$

subtract 2-dimensional coupled cat map lattice equation

$$-m_{nt} = \phi_{n,t+1} + \phi_{n,t-1} - 2s\,\phi_{nt} + \phi_{n+1,t} + \phi_{n-1,t}$$

cat herd is thus governed by the law of

***d*-dimensional spatiotemporal cat**

$$(-\Box + \mu^2)\,\phi_z = m_z\,, \qquad \mu^2 = d(s-2)$$

where $\phi_z \in [0,1)\,,\quad m_z \in \mathcal{A}$ and $z \in \mathbb{Z}^d$ = integer lattice

discretized linear PDE

d-dimensional spatiotemporal cat

$$(-\Box + \mu^2)\,\phi_z = m_z$$

is linear and known as

- tight-binding model or Helmholtz equation
  if stretching is weak, $s < 2$
  [oscillatory sine, cosine solutions]
- Euclidean Klein-Gordon or (damped Poisson)
  if stretching is strong, $s > 2$
  [hyperbolic sinches, coshes, ‘mass’ $\mu^2 = d(s-2)$]

nonlinearity is hidden in the ‘sources’

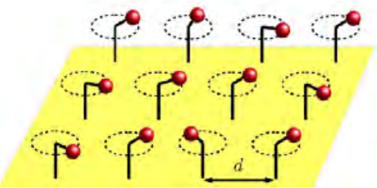
$$m_z \in \mathcal{A} \text{ at lattice site } z \in \mathbb{Z}^d$$

spring mattress vs field of rotors

traditional field theory

Helmholtz

chaotic field theory



damped Poisson

the simplest of all ‘turbulent’ field theories !

spatiotemporal cat

$$(-\Box + \mu^2)\,\phi_z = m_z$$

can be solved completely (?) and analytically (!)

assign to each site z a letter m_z from the alphabet $\mathcal{A}$.

a particular fixed set of letters m_z (a symbol block)

$$\mathsf{M} = \{m_z\} = \{m_{n_1 n_2 \cdots n_d}\}\,,$$

is a complete specification of
the corresponding lattice state X

from now on work in $d = 2$ dimensions, ‘stretching parameter’ $s = 5/2$

think globally, act locally

solving the spatiotemporal cat equation

$$\mathcal{J}\mathsf{X} = -\mathsf{M}\,,$$

with the $[n\times n]$ matrix $\qquad \mathcal{J} = \sum_{j=1}^{2}\left(\sigma_j - s\mathbb{1} + \sigma_j^{-1}\right)$

can be viewed as a search for zeros of the function

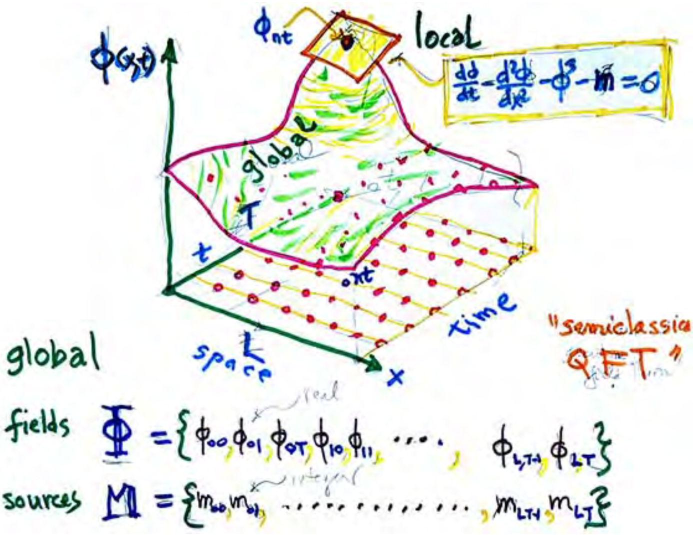
$$F[\mathsf{X}] = \mathcal{J}\mathsf{X} + \mathsf{M} = 0$$

where the entire global lattice state X_M is

a single fixed point $\mathsf{X}_\mathsf{M} = \{\phi_z\}$
in the LT-dimensional unit hyper-cube $\mathsf{X} \in [0,1)^{LT}$

L is the 'spatial', T the 'temporal' lattice period

think globally, act locally



for each symbol array M, a periodic lattice state X_M

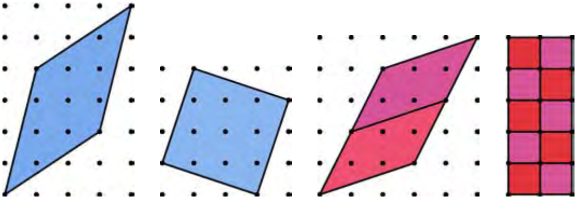
next, enumerate all periodic spacetime tilings of the integer lattice

each tile : 2-dimensional (sub)lattice, an infinite array of points

$$\Lambda = \{n_1\mathbf{a}_1 + n_2\mathbf{a}_2 \,|\, n_i \in \mathbb{Z}\}$$

with the defining tile spanned by a pair of basis vectors $\mathbf{a}_1, \mathbf{a}_2$

example : four tiles of area 10



The two blue tiles appear 'prime', i.e., not tiled by smaller tiles.
False!   all four big tiles can tilled by smaller ones.

tricky!

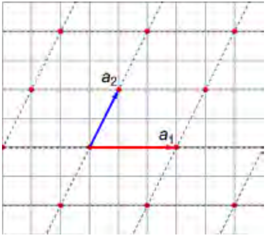
2-dimensional lattice tilings

2-dimensional *lattice* is defined by a $[2 \times 2]$ fundamental parallelepiped matrix whose columns are basis vectors

$$\mathbf{A} = [\mathbf{a}_1 \mathbf{a}_2] = \begin{bmatrix} L & S \\ 0 & T \end{bmatrix} ,$$

L, T : spatial, temporal lattice periods
'tilt' $0 \leq S < L$ imposes the *relative-periodic* (*'helical'*, *'toroidal'*, *'twisted'*, *'screw'*, $\cdots$) bc's

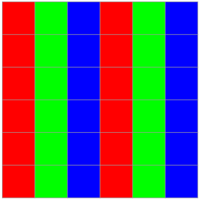
example : $[3\times 2]_1$ tile



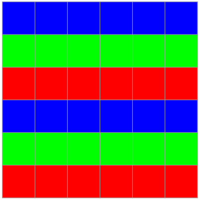
basis vectors

$$\mathbf{a}_1 = \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \mathbf{a}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

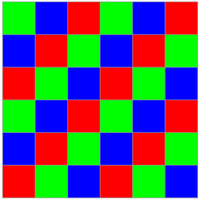
exponentially many periodic lattice states in Felinestan



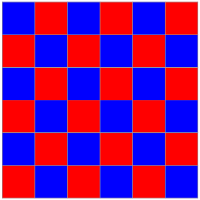
$[3\times1]_0$



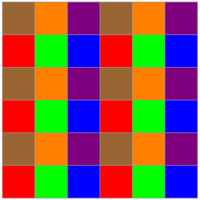
$[1\times3]_0$



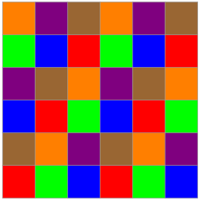
$[3\times1]_1$



$[2\times1]_1$



$[3\times2]_0$



$[3\times2]_1$

tile color = value of symbol m_z

note : spatiotemporal cat dances over a parquet floor

(so far) latticization of spacetime continuum :
field $\phi(x, t)$ over spacetime coordinates (x, t)
for *any* field theory

$$\Rightarrow$$

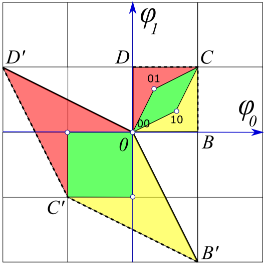
set of lattice site values $\phi_z = \phi(n\Delta L, t\Delta T)$.
Subscript $z = (n, t) \in \mathbb{Z}^d$ is a discrete d-dimensional spacetime *coordinate* over which the field ϕ lives

distinct spacetime tiles have tilted shapes $[L\times T]_S$

(next) spatiotemporal cat *field* ϕ_z is confined to $[0, 1)$
That imparts a $\mathbb{Z}^1$ lattice structure on fundamental parallelepiped $\mathcal{J}$ basis vectors ; fundamental fact then counts all periodic lattice *states* $\mathrm{X_M}$ for a given spacetime tile $[L\times T]_S$

fundamental fact works for a spacetime lattice (!)

recall Bernoulli fundamental fact example ?



unit hyper-cube $\mathsf{X} \in [0,1)^2$

$$\Rightarrow \mathcal{J} \Rightarrow$$

fundamental parallelepiped

spacetime fundamental parallelepiped basis vectors $\mathsf{X}^{(j)}$
$=$ columns of the orbit Jacobian matrix

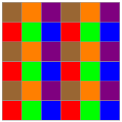
$$\mathcal{J} = (\mathsf{X}^{(1)}|\mathsf{X}^{(2)}|\cdots|\mathsf{X}^{(LT)})$$

example : spacetime periodic $[3\times2]_0$ lattice state

$$F[\mathsf{X}] = \mathcal{J}\mathsf{X} + \mathsf{M} = 0$$

6 field values, on 6 lattice sites $z = (n, t)$, $[3\times2]_0$ tile :

$$\mathsf{X}_{[3\times2]_0} = \begin{bmatrix} \phi_{01} & \phi_{11} & \phi_{21} \\ \phi_{00} & \phi_{10} & \phi_{20} \end{bmatrix} , \qquad 6\,\mathsf{M}_{[3\times2]_0} =$$



where the region of symbol plane shown is tiled by 6 repeats of the $\mathsf{M}_{[3\times2]_0}$ block, and tile color = value of symbol m_z

'stack up' vectors and matrices, vectors as 1-dimensional arrays,

$$\mathsf{X}_{[3\times2]_0} = \left(\begin{array}{c} \phi_{01} \\ \phi_{00} \\ \hline \phi_{11} \\ \phi_{10} \\ \hline \phi_{21} \\ \phi_{20} \end{array}\right) , \qquad \mathsf{M}_{[3\times2]_0} = \left(\begin{array}{c} m_{01} \\ m_{00} \\ \hline m_{11} \\ m_{10} \\ \hline m_{21} \\ m_{20} \end{array}\right)$$

with the $[6\times6]$ orbit Jacobian matrix in block-matrix form

$$\mathcal{J}_{[3\times2]_0} = \left(\begin{array}{cc|cc|cc} -2s & 2 & 1 & 0 & 1 & 0 \\ 2 & -2s & 0 & 1 & 0 & 1 \\ \hline 1 & 0 & -2s & 2 & 1 & 0 \\ 0 & 1 & 2 & -2s & 0 & 1 \\ \hline 1 & 0 & 1 & 0 & -2s & 2 \\ 0 & 1 & 0 & 1 & 2 & -2s \end{array}\right)$$

fundamental parallelepiped basis vectors $\mathsf{X}^{(j)}$ are the columns of the orbit Jacobian matrix

$$\mathcal{J}_{[3\times2]_0} = \left(\begin{array}{c|c|c|c|c|c} -2s & 2 & 1 & 0 & 1 & 0 \\ 2 & -2s & 0 & 1 & 0 & 1 \\ 1 & 0 & -2s & 2 & 1 & 0 \\ 0 & 1 & 2 & -2s & 0 & 1 \\ 1 & 0 & 1 & 0 & -2s & 2 \\ 0 & 1 & 0 & 1 & 2 & -2s \end{array}\right)$$

the ‘fundamental fact’ now yields the number of solutions for any half-integer s as Hill determinant

$$N_{[3\times2]_0} = |\mathrm{Det}\,\mathcal{J}_{[3\times2]_0}| = 4(s-2)s(2s-1)^2(2s+3)^2$$

can count spatiotemporal cat states for any $\Lambda = [L \times T]_S$

Λ	$N_\Lambda(s)$	$M_\Lambda(s)$
$[1\times 1]_0$	$2(s-2)$	$2(s-2)$
$[2\times 1]_0$	$2(s-2)2s$	$2(s-2)\frac{1}{2}(2s-1)$
$[2\times 1]_1$	$2(s-2)2(s+2)$	$2(s-2)\frac{1}{2}(2s+3)$
$[3\times 1]_0$	$2(s-2)(2s-1)^2$	$2(s-2)\frac{4}{3}(s-1)s$
$[3\times 1]_1$	$2(s-2)4(s+1)^2$	$2(s-2)\frac{1}{3}(2s+1)(2s+3)$
$[4\times 1]_0$	$2(s-2)8(s-1)^2 s$	$2(s-2)\frac{1}{2}(2s-3)(2s-1)s$
$[4\times 1]_1$	$2(s-2)8s^2(s+2)$	$2(s-2)\frac{1}{2}(s+2)(2s-1)(2s+1)$
$[4\times 1]_2$	$2(s-2)8(s+1)^2 s$	$2(s-2)\frac{1}{2}(2s+3)(2s+1)s$
$[4\times 1]_3$	$2(s-2)8s^2(s+2)$	$2(s-2)\frac{1}{2}(s+2)(2s-1)(2s+1)$
$[5\times 1]_0$	$2(s-2)\left(4s^2-6s+1\right)^2$	$2(s-2)\frac{4}{5}(s-1)(2s-3)(2s-1)s$
$[5\times 1]_1$	$2(s-2)16\left(s^2+s-1\right)^2$	$2(s-2)\frac{1}{5}(2s-1)(2s+3)(4s^2+4s-5)$
$[2\times 2]_0$	$2(s-2)8s^2(s+2)$	$2(s-2)\frac{1}{2}(2s-1)(2s^2+5s+1)$
$[2\times 2]_1$	$2(s-2)8s(s+1)^2$	$2(s-2)\frac{1}{2}(2s+1)(2s+3)s$
$[3\times 2]_0$	$2(s-2)2s(2s-1)^2(2s+3)^2$	$2(s-2)\frac{2}{3}(2s-1)(4s^3+10s^2+3s-5)s$
$[3\times 2]_1$	$2(s-2)32s^3(s+1)^2$	$2(s-2)\frac{1}{6}(2s-1)(2s+1)(8s^3+16s^2+10s+$
$[3\times 3]_0$	$2(s-2)16(s+1)^4(2s-1)^4$	

we can count !

1. can construct all spacetime tilings, from small tiles to as large as you wish
2. for each spacetime tile $[L \times T]_S$, can evaluate $\sharp$ of doubly-periodic lattice states for a tile

$$N_{[L \times T]_S}$$

3. $\sharp$ prime orbits for a tile

$$M_{[L \times T]_S}$$

zeta function for a field theory ???

‘periodic orbits’ are now invariant 2-tori (tiles)

each a spacetime lattice tile p of area $A_p = L_p T_p$
that cover the phase space with ‘natural weight’

$$\sum_p \frac{1}{|\mathrm{Det}\,\mathcal{J}_p|}$$

at this time :

- $d = 1$ temporal cat zeta function works like charm
- $d = 2$ spatiotemporal cat works, order by order
- $d \geq 2$ Navier-Stokes zeta is still but a dream

spatiotemporal cat topological zeta function

know how to evaluate the number of doubly-periodic lattice states

$$N_{[L\times T]_S}\,,$$

for a given $[L\times T]_S$ finite spacetime tile

now substitute these numbers of lattice states into the topological zeta function

$$1/\zeta_{\mathrm{top}}(z_1,z_2) \;=\; 1-\frac{\mu^2}{z_1+z_2-4+z_1^{-1}+z_2^{-1}}$$

??

but that's just a guess - we currently have no generating function that presents all solutions in a compact form

funky... not solved :(

2.15 Integer lattices literature

There are many reasons why one needs to compute an "orbit Jacobian matrix" Hill determinant $|\mathrm{Det}\,\mathcal{J}|$, in fields ranging from number theory to engineering, and many methods to accomplish that:

discretizations of Helmholtz [58] and screened Poisson [59, 80, 96, 97] (also known as Klein–Gordon or Yukawa) equations

Green's functions on integer lattices [5, 8, 24, 33, 37, 40, 63, 67, 78, 92, 93, 115–117, 135, 140, 143, 149, 150, 159, 180, 196]

Gaussian model [71, 111, 139, 172]

linearized Hartree-Fock equation on finite lattices [121]

quasilattices [29, 69]

circulant tensor systems [33, 37, 146, 164, 166, 200]

Ising model [19, 88, 89, 98, 100, 103–105, 128, 136, 141, 153, 161, 199], transfer matrices [154, 199]

lattice field theory [108, 144, 148, 151, 168, 175, 176, 192]

modular transformations [34, 205]

lattice string theory [77, 157]

random walks, resistor networks [9, 25, 49, 50, 60, 81, 86, 99, 122, 163, 183, 188, 198]

spatiotemporal stability in coupled map lattices [4, 75, 203]

Van Vleck determinant, Laplace operator spectrum, semiclassical Gaussian path integrals [47, 125, 126, 187]

Hill determinant [26, 47, 137]; discrete Hill's formula and the Hill discriminant [186]

Lindstedt-Poincaré technique [189–191]

heat kernel [38, 61, 64, 110, 114, 143, 159, 201]

lattice points enumeration [15, 16, 20, 56]

primitive parallelogram [10, 30, 152, 193]

difference equations [55, 68, 181]

digital signal processing [62, 130, 197]

generating functions, Z-transforms [64, 194]

integer-point transform [20]

graph Laplacians [41, 79, 134, 162]

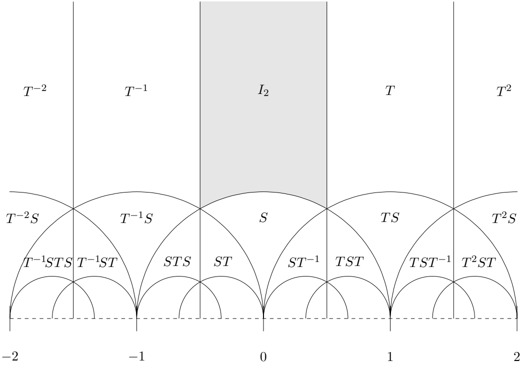
graph zeta functions [7, 13, 18, 27, 42–44, 57, 61, 83, 87, 94, 101, 123, 124, 162, 165, 169, 171, 179, 184, 185, 204]

zeta functions for multi-dimensional shifts [12, 132, 133, 147]

zeta functions on discrete tori [38, 39, 201]

side remark to experts

$SL_2(\mathbb{Z})$ unimodular invariance of square lattice



Action on the complex upper half-plane by linear fractional transformations T and S. (figure: Keith Conrad)

infinitely many 'Bravais cells' for the same lattice

but, is this

chaos?

yes, short tiles are exponentially good ‘shadows’ of the larger ones, so can attain any desired accuracy

is spatiotemporal cat ‘chaotic’?

in time-evolving deterministic chaos any chaotic trajectory is shadowed by shorter periodic orbits

in spatiotemporal chaos, any unstable lattice state is shadowed by smaller invariant 2-tori (Gutkin *et al.*[3,4])

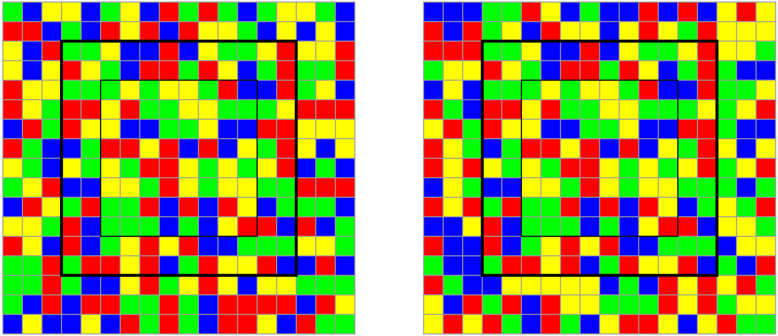
next figure : code the M symbol block ϕ_{nt} at the lattice site *nt* with (color) alphabet

$$m_{t\ell} \in \mathcal{A} = \{\underline{1}, 0, 1, 2, \cdots\} = \{\textit{red}, \textit{green}, \textit{blue}, \textit{yellow}, \cdots\}$$

[3] B. Gutkin and V. Osipov, Nonlinearity **29**, 325–356 (2016).

[4] B. Gutkin et al., Nonlinearity **34**, 2800–2836 (2021).

shadowing, symbolic dynamics space

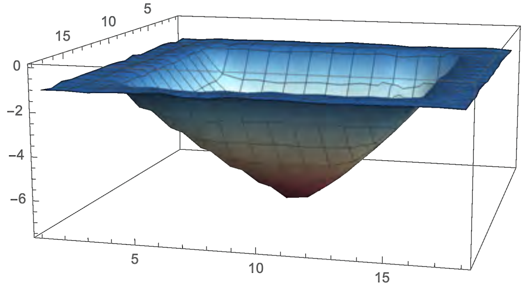


2d symbolic representation M_j of two lattice states X_j shadowing each other within the shared block $\mathsf{M}_{\mathcal{R}}$

- border $\mathcal{R}$ (thick black)
- symbols outside $\mathcal{R}$ differ

$s = 7/2$ Adrien Saremi 2017

shadowing



the logarithm of the average of the absolute value of site-wise distance

$$\ln |\phi_{2,z} - \phi_{1,z}|$$

averaged over 250 solution pairs

note the exponential falloff of the distance away from the center of the shared block $\mathcal{R}$

$\Rightarrow$ within the interior of the shared block,

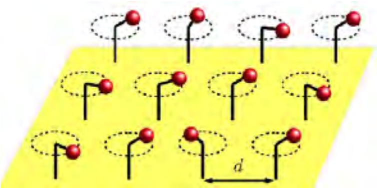
shadowing is exponentially close

take-home :

harmonic field theory

tight-binding model
(Helmholtz)

chaotic field theory



Euclidean Klein-Gordon
(damped Poisson)

our song of chaos has been sang. what next ?

1. coin toss
2. kicked rotor
3. spatiotemporal cat
4. bye bye, dynamics