2-Dimensional Inelastic Impact Osciallator

Tyson R. Shepherd
gtg084n@mail.gatech.edu
Department of Physics
Georgia Institute of Technology

December 4, 2003

Abstract

We present an investigation of a two-dimensional inelastic impact oscillator representing a friction force microscope. This paper includes the equations of motion, an analysis of the Jacobi matrix, and the fundamental domain. By applying periodic orbit theory to these and then calculating the diffusion, we find a possible link between nanotribology and everyday kinetic friction. Included is a simulation, trajectory plots, and Poincareḿaps.

1 Introduction

An impact oscillator is a system which has a forcing function driving a particle to repeatedly impact a barrier. The impact oscillator has been an important system to study for both engineers and physicists. For engineers, it has such applications as earthquake vibrations [1], articulated mooring towers [2], and engine rattling [3]. Fermi investigated the same system when dealing with cosmic rays impacting the atmosphere [4]. The impact osciallator is also ideal for studying the dynamics of a seemingly chaotic system: it is conceptually easy to understand, yet there is still much to learn about it.

One particular generalization of the impact oscillator is to consider a particle bouncing on a two dimensional, infinitely long, infinitely dense table with some known periodic defect or curvature. The particle is moving at some velocity and there exists a potential due to a combination of a vertical and horizontal springs acting on the particle. The ball has some coefficient of restitution that results in energy loss at the barrier interaction. There may also be energy loss tangent to the surface [5] due to some friction which will resist the translational motion of the ball along the surface.

This model is important to study because of its inherent connection to microscopic friction. Although the Tomlinson model [6] is the most widely used for approximations to the friction force microscope (FFM), we use the more realistic Gyalog-Thomas model [7]. In this paper we model an FFM as it moves across a surface with regular defects with a velocity large enough to bounce chaotically off the surface. The parameters chosen for the simulation are in accordance with what can be attainable in the labratory. The natural frequency for the verticle spring is $\omega_y = \sqrt{\frac{k_y}{m}} = 35\ kHz$, the horizontal spring's natural frequency is given by $\omega_x = \sqrt{\frac{k_x}{m}} = 442\ kHz$. Drift velocities will be bounded by $1\ \frac{nm}{s}\ <\ v_c\ <\ 50\ \frac{\mu m}{s}$.

The goal of this research is to study the dynamics of the system described above. This project will include an investigation of what velocities the chaotic bouncing will occur, both theoretically and numerically by using tools such as

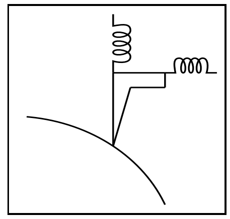

Figure 1: Simplified Atomic Force Microscope

bifurcation diagrams with the varied parameter being velocity. The impact oscillator will be recast into a billiard system and will then readily admit a Jacobian which implies a 3-dimensional Poincare' map can be found. The project will also include an investigation of the system using periodic orbit theory. A few additional facets to play with is using a Lorentz map of the maximum height the ball reaches relative to the surface verses the previous maximum height. This work will be a natural extention to the work done by Berg and Briggs [8].

The ultimate, and therefore ideal, goal of the project is to understand The impact system well enough to make a prediction of the coefficient of kinetic friction using deterministic diffusion.

2 Equations and Theory

2.1 Above the Table

We first consider the table to be composed of a sinusoidal curvature given by:

$$C = A * \sin(\frac{2\pi}{a}x)\ . \tag{1}$$

Where A is the scanning height and a is the lattice spacing. Now, with velocity given by v_c, potential by U, spring constants k, and mass m, the potential of the system while above the table is given by:

$$\begin{aligned} U_x &= \frac{1}{2} * k_x * (x - v_c t)^2 \\ U_y &= \frac{1}{2} * k_y * (y - Y_e)^2 \end{aligned} \tag{2}$$

Neglecting gravity, the forces on the particle while above the table are given by:

$$m\ddot{x} = -k_x x + k_x vt$$
$$m\ddot{y} = -k_y y + k_y Y_e$$

It is easily seen that the horizontal trajectory is given by:

$$x(t) = A_x \ \sin(\omega_x \ t) + B_x \ \cos(\omega_x \ t) + vt$$
$$\dot{x}(t) = A_x \ \omega_x \ \cos(\omega_x \ t) - B_x \ \omega_x \ \sin(\omega_x \ t) + v \tag{3}$$

While the verticle trajectory is given by:

$$y(t) = A_y \ \sin(\omega_y \ t) + B_y \ \cos(\omega_y \ t) + Y_e$$
$$\dot{y}(t) = A_y \ \omega_y \ \cos(\omega_y \ t) - B_y \ \omega_y \ \sin(\omega_y \ t) \tag{4}$$

Knowing that the particle will be at a position x_i with velocity v_{xi}^+ and y_i with velocity v_{yi}^+ after it impacts the table, the initial conditions for each bounce will be:

$$x(0) = x_i = B_x$$
$$\dot{x}(0) = v_{xi}^+ = A_x \ \omega_x + v$$
$$y(0) = y_i = B_y + Y_e$$
$$\dot{y}(0) = v_{xi}^+ = A_y \ \omega_y$$

Solving for A_x, B_x, A_y, B_y, and plugging these into 3 and 4 we have:

$$x(t) = \frac{(v_{xi}^+ - v)}{\omega_x} \ \sin(\omega_x \ t) + x_i \ \cos(\omega_x \ t) + vt \tag{5}$$
$$y(t) = \frac{v_{yi}^+}{\omega_y} \ \sin(\omega_y \ t) + (y_i - Y_e) \ \cos(\omega_y \ t) + Y_e \tag{6}$$

2.2 Impact

We now consider the equations of motion when the ball impacts the table. Let ϕ be the angle of the line perpendicular to the curvature of the table, with respect to the x-axis. First rotate the axis by an angle $\phi - \frac{\pi}{2}$ Then apply the two damping effects at impact: one due to a coefficient of restitution of the particle, γ, and the tangential damping due to kinetic friction, μ_k. These will show up as $v_y^{'+} = -\gamma \; v_y^{'-}$ and $v_x^{'+} = -\mu_k \; v_x^{'-}$. We then rotate the coordinates back to the original. We then have the following transform:

$$\begin{pmatrix} v_x^+ \\ v_y^+ \end{pmatrix} = \begin{pmatrix} \sin(\phi) & \cos(\phi) \\ -\cos(\phi) & \sin(\phi) \end{pmatrix} \begin{pmatrix} \mu_k & 0 \\ 0 & -\gamma \end{pmatrix} \begin{pmatrix} \sin(\phi) & -\cos(\phi) \\ \cos(\phi) & \sin(\phi) \end{pmatrix} \begin{pmatrix} v_x^- \\ v_y^- \end{pmatrix} \quad (7)$$

In the simulation $mu_k = 1$. This leads us to the following tranformation:

$$\begin{pmatrix} v_x^+ \\ v_y^+ \end{pmatrix} = \begin{pmatrix} -\gamma \; \cos^2(\phi) + \sin^2(\phi) & -(1+\gamma) \; \cos(\phi) \; \sin(\phi) \\ -(1+\gamma) \; \cos(\phi) \; \sin(\phi) & -\gamma \; \sin^2(\phi) + \cos^2(\phi) \end{pmatrix} \begin{pmatrix} v_x^- \\ v_y^- \end{pmatrix} \quad (8)$$

If the velocity is tangential enough or small enough to not leave the table, the tip will drag along the surface as is the normal use of the atomic force microscope.

2.3 The Jacobian

All the necessary tools are in our bag to find the Jacobian Matricies of our system. There are two important matricies to find. The first is for the motion while the particle is in transit above the table, which will be refered to as $\mathcal{J}_T$. The second describes the affects of the impact on the particle, which is called $\mathcal{J}_R$. The latter shall be described first, as it requires intuition, rather than mathematical riggor.

The foundations for the theory to follow are based on the principles defined here. The particle has a coefficience of restitution, $\gamma <= 1$ which is the factor by which the velocity is damped normal to the surface. Let the phase coordinates be chosen as $\vec{x} = (x', z, p'_x, p_z)$. For ease of calculations, the mass is set equal to 1. Let $x' = x - v_c \; t$, with x as defined above, putting us in the frame of the moving support. Let $z = y - Y_e - T$, where z is then the height above the table, with respect to the equalibrium of the FFM. $v_x; \; v_z$ are their respective momenta.

As the simplest example, consider a flat, motionless table, and two particles bouncing vertically off of it. There is a variation of position along the flow, normal to the flow, and there is also a small variation of their velocities along the flow. The first particle impacts the table at some time τ_1, and slows down to a velocity $v_z^+ = -\gamma v_z^-$. The second particle continues at v_z^- until it impacts the table at time τ_2, slowing down to $v_z^+ = -\gamma v_z^-$. We now know $\delta v_z^+ = -\gamma \delta v_z^-$. The difference in time,

$$\Delta\tau = \tau_2 - \tau_1 = \frac{\delta z^-}{\delta v_z^-}$$

Which implies

$$\delta z^+ = \frac{\delta z^-}{\delta v_z^-} * \delta v_z^+$$

but since $\delta v_z^+ = -\gamma \delta v_z^-$, we have $\delta q_z^+ = -\gamma \delta q_z^-$.

Now take the particles to impact the table at some angle ϕ. We take the velocity tangent to the surface to remain constant. Because the velocity stays in the same direction and the ordering of two nearby points remains the same, the matrix equation is given by:

$$\begin{pmatrix} \delta x^+ \\ \delta v_x^+ \\ \delta z^+ \\ \delta v_z^+ \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\gamma & 0 \\ 0 & 0 & 0 & -\gamma \end{pmatrix} \begin{pmatrix} \delta x^- \\ \delta v_x^- \\ \delta z^- \\ \delta v_z^- \end{pmatrix}$$

the Jacobian matrix for reflection off a flat table at some angle ϕ is then given by:

$$\mathcal{J}_R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\gamma & 0 \\ 0 & 0 & 0 & -\gamma \end{pmatrix}$$

Note that the Hamiltonian of a particle attached to a spring in the y and x directions, yet uncoupled is given by:

$$\mathcal{H}_x = \frac{p_x^2}{2m} + \frac{1}{2} m\omega_x^2 (x - v_c\ t)$$
$$\mathcal{H}_y = \frac{p_y^2}{2m} + \frac{1}{2} m\omega_y^2 (y - Y_e)$$

Transforming these to the coordinates designated above, the solution to our equations of motion are:

$$
\begin{aligned}
x^- &= x^+ \cos(\omega_x t) + \frac{v_x^+}{\omega_x} \sin(\omega_x t) \\
v_x^- &= -\omega_x x^+ \sin(\omega_x t) + v_x^+ \cos(\omega_x t) \\
z^- &= z^+ \cos(\omega_y t) + \frac{v_z^+}{\omega_y} \sin(\omega_y t) - A \sin(\frac{2\pi}{a}(x^- + v_c t) \\
v_z^- &= -\omega_y z^+ \sin(\omega_y t) + v_z^+ \cos(\omega_y t) - \frac{2 * \pi * A}{a}((v_x^- + v_c) \cos(\frac{2\pi}{a}(x^- + v_c t))
\end{aligned}
$$

Taking the infintesimal variations, setting m=1, with τ as the time of flight, and small angle approximations then gives:

$$
\begin{aligned}
\delta x^- &= \delta x^+ \cos(\omega_x \tau) + \delta v_x^+ \frac{\sin(\omega_x \tau)}{\omega_x} \\
\delta v_x^- &= -\omega_x \delta x^+ \sin(\omega_x \tau) + \delta v_x^+ \cos(\omega_x \tau) \\
\delta z^- &= \delta z^+ \cos(\omega_y \tau) + \delta v_z^+ \frac{\sin(\omega_y \tau)}{\omega_y} - \frac{2 * \pi * A}{a}(\delta x^+ \cos(\omega_x \tau) + \delta v_x^+ \frac{\sin(\omega_x \tau)}{\omega_x}) \\
\delta v_z^- &= -\omega_y \delta z^+ \sin(\omega_y \tau) + \delta v_z^+ \cos(\omega_y \tau) - \frac{2 * \pi * A}{a}(-\omega_x x^+ \sin(\omega_x \tau) + v_x^+ \cos(\omega_x \tau))
\end{aligned}
$$

This implies a transit Jacobian like:

$$
\mathcal{J}_T = \begin{pmatrix}
\cos(\omega_x\ \tau) & \frac{\sin(\omega_x\ \tau)}{\omega_x} & 0 & 0 \\
-\omega_x\ \sin(\omega_x\ \tau) & \cos(\omega_x\ \tau) & 0 & 0 \\
-\frac{2\pi A}{a} \cos(\omega_x\ \tau) & -\frac{2\pi A}{a} \frac{\sin(\omega_x\ \tau)}{\omega_x} & \cos(\omega_y\ \tau) & \frac{\sin(\omega_y\ \tau)}{\omega_y} \\
\frac{2\pi A}{a}\ \omega_x\ \sin(\omega_x\ \tau) & -\frac{2\pi A}{a} \cos(\omega_x\ \tau) & -\omega_y\ \sin(\omega_y\ \tau) & \cos(\omega_y\ \tau)
\end{pmatrix}
$$

Note that taking ω_y and ω_x to be tiny (i.e. free particles) this reduces to a nice free particle Jacobian. Also note that for $v_c = 0$ and $\gamma = 1$ the determinant of the Jacobian matricies are 1.

Another possibility of phase coordinates is to choose $x = (q_{||}, q_T, p_{||}, p_T)$. For ease of calculations, the mass is set equal to 1. The variation is then given by $\delta x = (\delta q_{||}, \delta q_T, \delta v_{||}, \delta v_T)$, where $\delta q_{||}$ is the variation along the flow, δq_T is the variation normal to the flow, and $\delta v_{||}, \delta v_T$ are their respective momenta.

Recall the analysis of the first example of a ball impacting a motionless flat table. Now take the particles to impact the table at some angle ϕ. From simply geometry we see that

$$\frac{\Delta z}{\sin(\phi)} = \frac{\delta q_T^-}{\sin(\frac{\pi}{2} - \phi)} = \frac{\Delta x}{\sin(\frac{\pi}{2})}$$

which implies that $\Delta z = \delta q_T^- \ \tan(\phi)$. Now

$$\delta q_{||}^+ = \frac{\Delta z \ (\delta v_{||}^- - \delta v_{||}^+)}{\delta v_{||}^-} = \Delta z \ (1-\gamma) = \delta q_T^- \ (1-\gamma) \ \tan(\phi)$$

Which gives:

$$\delta q_{||}^+ = \gamma \delta q_{||}^- + \delta q_T^- \ (1-\gamma) \ \tan(\phi)$$

The Jacobian matrix for reflection off a flat table at some angle ϕ is then given by:

$$\mathcal{J}_R = \begin{pmatrix} \gamma & 0 & (1-\gamma)\tan(\phi_k) & 0 \\ 0 & \gamma & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

More analysis is left for future study.

2.4 The Fundamental Domain

A nice fundamental domain is always an important tool for cyclists, if it can be found. It is an axiom of this project that the defects in the surface are periodic. This allows for a natural fundamental domain to be defined as a the surface as the bottom boundary, and two walls which preserve energy to be bounding period of the of the defect. It may even be reduced more depending on the specific defect. Consider the table composed of a sinusoidal defect, as in 1. The elastic walls will then be placed at x´ = 0 and $\frac{\pi}{2\ a}$. There will be no boundry at the top. This setup will have the same dynamics as that of the original system. See 2. Sadly, because there is springs working in each direction, there are not pruning rules for the setup.

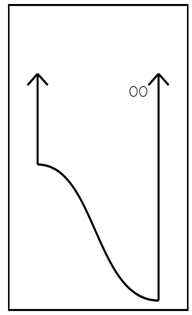

Figure 2: The Fundamental Domain

2.5 Deterministic De-friction?

2.6 The Simultation

Appendix 1 is a listing of the simulation code for the trajectories. Pictured are the results from the simulation using the equations derived above. Figures 5 a,b are of two nearby trajectories that give vastly different results as time goes on. Figure 6 a,b are Poincareḿaps with the snapshots taken when $v_y = 0$ and at the impact of the table, respectively. Figure 7 is a Lorentz map of when $v_y = 0$.

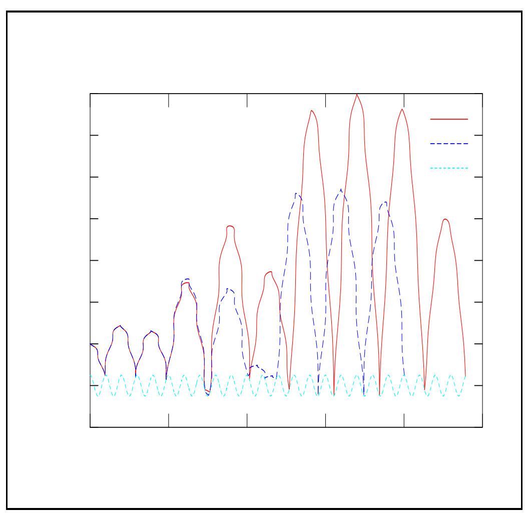

Figure 3: Nearby Initial Condition separate rapidly. Is this chaos?

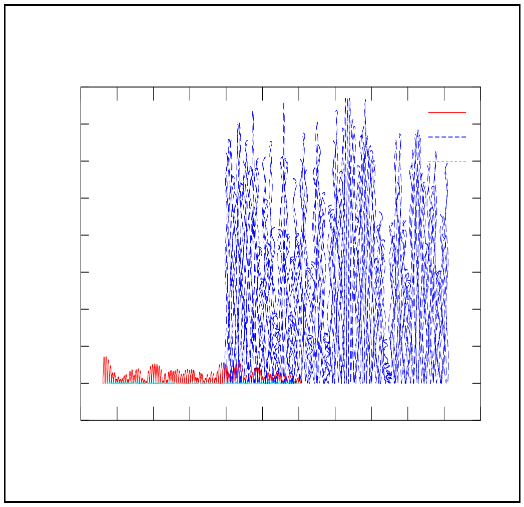

Figure 4: Vastly different behavior in the long run.

References

[1] J. M. T. Thompson and R. Ghaffari. *Phys. Lett. A* **91**, 5 (1982).

[2] J.M.T. Thompson, A.R. Bokaian and R. Ghaffari. *J. Energy Resources Technology (Trans ASME)*, **106**, 191-198 (1984).

[3] S. W. Shaw and P. J. Holmes. *Phys. Rev. Lett.* **51**, 623 (1983).

[4] E. Fermi. *Phys. Rev.* **75**, 1169 (1949).

[5] J. P. Cleveland, B. Anczykowski, A. E. Schmid, and V. B. Elings. *Appl. Phys. Lett.* **72**, 2613 (1998).

[6] G. A. Tomlinson, *Philos. Mag* **7**, 905 (1929).

[7] T. Gyalog and H. Thomas, Z. *Phys. Lett. B* **104**, 669 (1997).

[8] J. Berg and G. A. D. Briggs. *Phys. Rev. B* **55**, 14899 (1997).

[9] P. Cvitanovic, *et al. Classical and Quantum Chaos.* Version 10, 2003. Available online from: http://www.nbi.dk/ChaosBook/ChaosBook