Part II

Two dimensional maps

Chapter 3

Two dimensional folding maps

The symbolic dynamics for the one dimensional maps has a solid mathematical basis [147, 152, 153, 175], and most statements can be proven. Part I introduced a new way to present bifurcations in a symbolic parameter space and gave some examples of calculation of statistical averages by using this symbolic description. When we want to study the symbolic dynamics and the bifurcations of two dimensional pruned maps, our basis is no longer mathematically proven theorems, but conjectures and intuition. The intention with this work on the pruned horseshoe maps is not to give rigorous proofs of the theory, but to show that the theory works and gives several interesting results, e.g. a systematic bifurcation diagram of the Hénon map which has not been obtained before.

To study the symbolic description of a pruned horseshoe like the Hénon map we have to combine the symbolic description of the complete Smale horseshoe [184] and the methods we used discussing the one dimensional n–modal maps. Using this we can obtain bifurcation diagrams for not complete horseshoe maps and a description of the non-wandering set. One way to understand the pruned horseshoe map is to describe its pruning front as done by Cvitanović, Gunaratne and Procaccia [53]. An equivalent description consists of finding a systematic approximation of the horseshoe by one dimensional maps. The later approach is more convenient if we want to find bifurcation diagrams but we discuss the close relation between the two points of view. We will also use the pruning front technique to describe Hamiltonian billiard systems. We will first discuss the horseshoe systems and the results obtained by Smale [184] and discuss the ordering of symbols in different horseshoe maps.

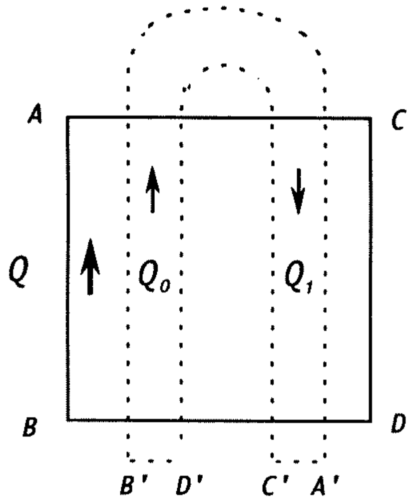


Figure 3.1: The Smale horseshoe map. The function g maps the square Q into the horseshoe.

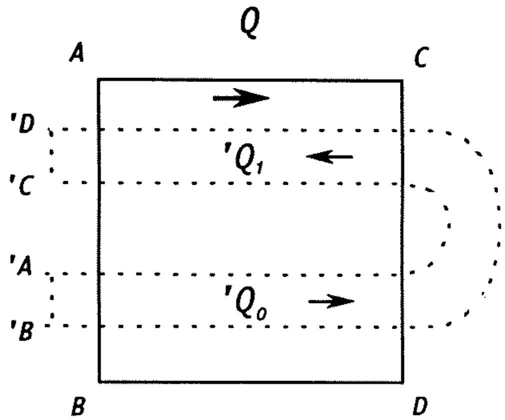


Figure 3.2: The inverse mapping g^{-1} of the Smale horseshoe.

3.1 The Smale horseshoe

The simplest example of a structurally stable chaotic diffeomorphism is the horseshoe map defined by Smale in 1961 [183]. Figure 3.1 shows the construction of this set as defined in ref. [184]. Q is a square in $\mathbb{R}^2$ drawn with solid lines in figure 3.1, mapped by the map g to the area bounded by the dashed lines. The map g is a diffeomorphism on Q and maps the corners $A \to A'$, $B \to B'$, $C \to C'$ and $D \to D'$, and g is a linear map on each component of $g^{-1}(g(Q) \cap Q)$. We denote the intersections $g(Q) \cap Q$ as Q_k with $k \in \{0, 1\}$. The index k is chosen such that when $x \in Q$ moves from the bottom to the top then $g(x)$ run through Q_k with increasing k. Figure 3.2 shows the action of g^{-1} on the square Q. The map g^{-1} maps the corners $A \to {}'A$, $B \to {}'B$, $C \to {}'C$ and $D \to {}'D$. We let ${}'Q_k$ be the two intersections $g^{-1}(Q) \cap Q$ such that $g({}'Q_k) \subset Q_k$.

We are interested in the subset Ω of Q where Ω is the non-wandering set of Q. A point $x \in Q$ is a wandering point if there is a neighborhood U of x such that $\bigcup_{|m|>0} g^{(m)}(U) \cap U = \emptyset$, and a point is called non-wandering if it is not a wandering point. Ω is the union of all the non-wandering points. Define Λ to be the intersection of all images and preimages of Q.

The following propositions are proved by Smale [184]

Proposition I. *The subset Λ of Q is compact, invariant under g, indecomposable and on Ω, g is topologically conjugate to the shift automorphism $\sigma : X_S \to X_S$,* *with the cardinality of* $S = 2$.

PROPOSITION II. *For a perturbation* g' *of* g, Λ' *defined similarly is also compact and invariant under* g'. *Then* $g' : \Lambda' \to \Lambda'$ *is also topologically conjugate to the shift* $\sigma : X_S \to X_S$.

From this follows that the non-wandering set of the Smale horseshoe can be described by a binary symbolic dynamics. A point $x \in \Lambda$ is mapped into a bi-infinite symbol string

$$\ldots s_{-2}s_{-1}s_0 \cdot s_1 s_2 \ldots \qquad s_i \in \{0,1\} \tag{3.1}$$

by choosing

$$s_i = k \quad \text{if} \quad g^{(i)}(x) \in Q_k \tag{3.2}$$

with $k \in \{0,1\}$ and $i \in Z$. For horseshoe maps with n folding we generalize to $k \in \{0,1,\ldots,n\}$. The iteration $x_t \to x_{t+1} = g(x_t)$ is in the symbolic description the shift

$$\sigma(\ldots s_{-2}s_{-1}s_0 \cdot s_1 s_2 \ldots) = \ldots s_{-2}s_{-1}s_0 s_1 \cdot s_2 \ldots \tag{3.3}$$

This is similar to the shift in the unimodal one dimensional map, but in the one dimension the past symbols are thrown away. Since the horseshoe is a diffeomorphism we need to know both the future and the past.

In figure 3.1 we see that the interaction Q_0 is oriented the same way as Q, while Q_1 is turned around and oriented opposite to Q. This gives the same ordering of the future symbols as for the unimodal map with a maximum point, because the change of orientation of Q_1 corresponds to the negative slope $f'(x) < 0$ for $x > x_c$ in the unimodal map. The well ordered future symbols w_i and the future symbolic value γ are obtained from (1.18)

$$\begin{aligned} w_1 &= s_1 \\ w_{t+1} &= \begin{cases} w_t & \text{if } s_{t+1} = 0 \\ 1 - w_t & \text{if } s_{t+1} = 1 \end{cases} \\ \gamma &= 0.w_1 w_2 w_3 \ldots = \sum_{t=1}^{\infty} \frac{w_t}{2^t} \end{aligned} \tag{3.4}$$

This symbolic value γ corresponds to the position of the coordinate x_t in the vertical direction in figure 3.1. The points $x_t \in \Lambda$ closest to the line BD have $\gamma = 0.0$ and the points closest to the line AC have $\gamma = 1.0$.

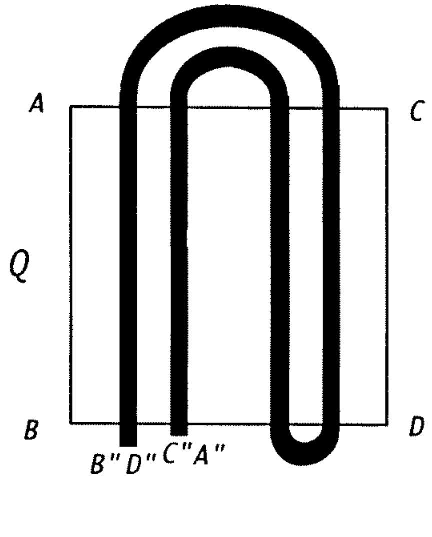


Figure 3.3: The image $g^{(2)}(Q)$ obtained by applying the Smale horseshoe map twice.

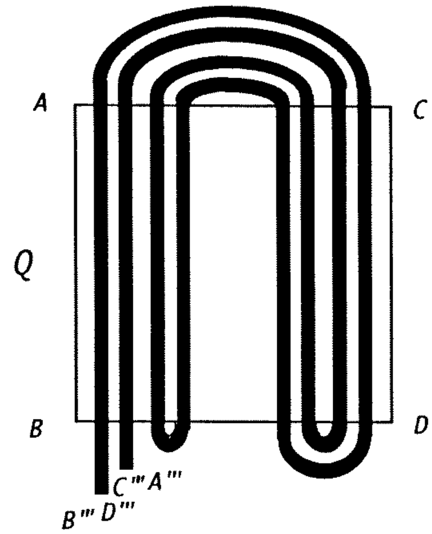


Figure 3.4: The image $g^{(3)}(Q)$ obtained by applying the Smale horseshoe map three times.

Figures 3.3 and 3.4 are the images obtained by applying the horseshoe map two times and three times on Q. The intersections $g^{(2)}(Q) \cap Q$ are 4 rectangles and $g^{(3)}(Q) \cap Q$ are 8 rectangles.

In figure 3.5 the square Q is drawn together with the 2^2 squares of $\Lambda_1 = g^{(-1)}(Q) \cap g^{(1)}(Q)$ and the 2^4 squares of $\Lambda_2 = g^{(-2)}(Q) \cap g^{(2)}(Q)$. The y–axis in figure 3.5 is labeled by $\cdot s_1 s_2$ and the x–axis by $s_{-1} s_0 \cdot$ and they give a unique labeling of the 2^4 squares of Λ_2. The n-th generation of the construction of the non-wandering Cantor set gives 2^{2n} squares of the set $\Lambda_n = g^{(-n)}(Q) \cap g^{(n)}(Q)$. From proposition II we know that this is also valid for a perturbation g' and the 2^{2n} parts of the perturbed intersections Λ'_n are called rectangles.

The picture of the inverse map in figure 3.2 shows that $'Q_0$ has the same orientation as Q while $'Q_1$ has opposite orientation. Well ordered symbols for the past are then obtained as for the unimodal map with a maximum point ($f'(x) < 0$ for $x < x_c$ and $f'(x) > 0$ for $x > x_c$) and the symbolic value δ for the past is obtained by the algorithm

$$
\begin{aligned}
w_0 &= s_0 \\
w_{t-1} &= \begin{cases} w_t & \text{if } s_{t-1} = 0 \\ 1 - w_t & \text{if } s_{t-1} = 1 \end{cases} \quad . \\
\delta &= 0.w_0 w_{-1} w_{-2} \ldots = \textstyle\sum_{t=1}^{\infty} \frac{w_{1-t}}{2^t}
\end{aligned}
\tag{3.5}
$$

The value δ increases along the horizontal position of x_t in figure 3.1. The points

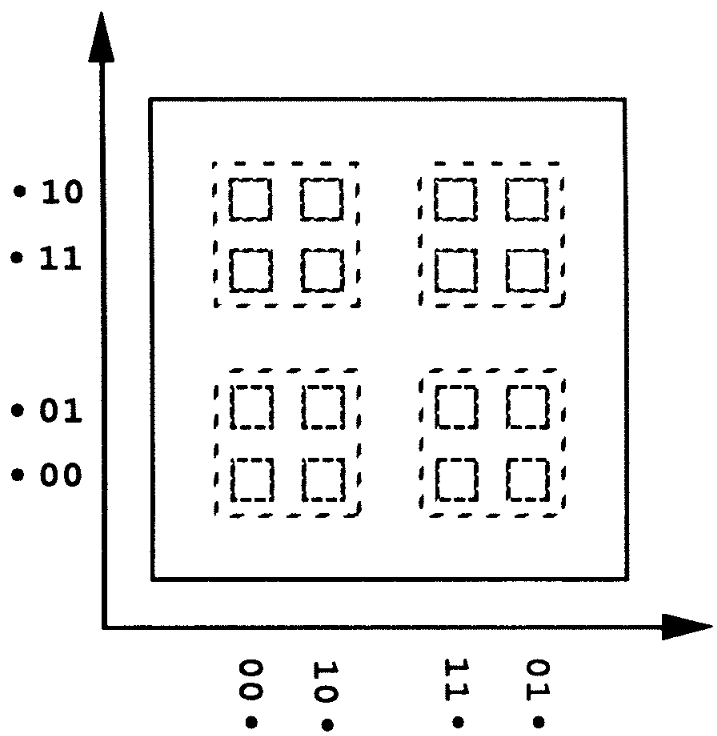


Figure 3.5: The square Q and the sets Λ_1 and Λ_2 of the Smale horseshoe and the labels $\cdot s_1 s_2$ and $s_{-1}s_0\cdot$.

$x_t \in \Lambda$ closest to the line AB have $\delta = 0.0$ and the points closest to the line CD have $\delta = 1.0$. The symbolic coordinate (δ, γ) gives a position in the Cantor set of figure 3.5 with the gaps removed. We call this coordinate the point in the symbol plane for the phase space point x.

3.1.1 Smale horseshoe with reflection

By adding a reflection around the vertical axis before applying the horseshoe map g we get the map $\tilde{g}$ showed in figure 3.6 and the inverse map $\tilde{g}^{-1}$ drawn in figure 3.7.

From the figures we find that $\tilde{Q}_0$ and $\tilde{Q}_1$ are oriented as Q_0 and Q_1 and the definition of the future symbolic value γ is identical to the not reflected horseshoe. The inverse intersections $'\tilde{Q}_0$ and $'\tilde{Q}_1$ are oriented such that $'\tilde{Q}_0$ is opposite to Q while $'\tilde{Q}_1$ has the same orientation as Q. This give an algorithm for the past symbolic value δ identical to the algorithm for a unimodal map with a minimum point such that the well ordered symbols changes with $s_t = 0$;

$$\begin{aligned} w_0 &= s_0 \\ w_{t-1} &= \begin{cases} 1 - w_t & \text{if } s_{t-1} = 0 \\ w_t & \text{if } s_{t-1} = 1 \end{cases} \quad . \\ \delta &= 0.w_0 w_{-1} w_{-2} \ldots = \sum_{t=1}^{\infty} \frac{w_{1-t}}{2^t} \end{aligned} \tag{3.6}$$

Figure 3.10 shows the set Λ_2 and the labels on the squares.

Two and three applications of $\tilde{g}$ give the folding in figures 3.8 and 3.9.

The Smale horseshoe with and without reflection is closely related and both may be realized by the Hénon map.

The shift operation σ on the symbols s_t in eq. (3.3) becomes more complicated when acting on well ordered symbols w_t. This type of shift operations are discussed for one example by Troll in ref. [193]. A shift operation shift the symbol string to the left but will also change $w_t \to 1 - w_t$ if a symbol that changes the ordering are moved from the future to the past symbolic description.

3.2 Variations of the Smale horseshoe

3.2.1 Once-folding maps

Smale showed that variations of the horseshoe map like figures 3.11, 3.13, 3.15 and 3.21 also yield non-wandering sets. The horseshoe in figure 3.11 is binary but the orientation is different than in figure 3.1. Both Q_0 and Q_1 are oriented the same way as Q and the construction of well ordered symbols is therefore simpler. The

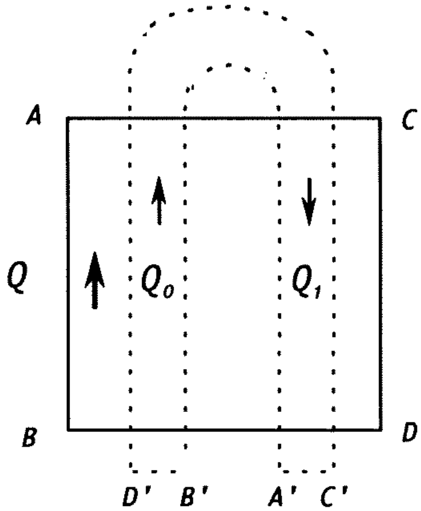


Figure 3.6: The Smale horseshoe map with reflection. The function $\tilde{g}$ maps the square Q into the horseshoe.

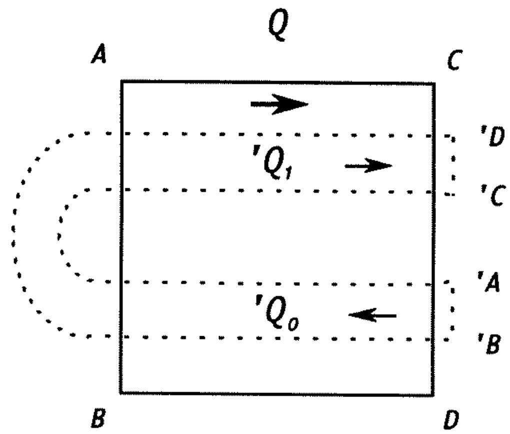


Figure 3.7: The inverse mapping $\tilde{g}^{-1}$ of the Smale horseshoe with reflection.

symbols w_t is constructed as for a one dimensional map with two branches, both having $f'(x) > 0$ like the Bernoulli shift $x_{t+1} = 2x_t \bmod (1)$. This gives simply

$$w_t = s_t \tag{3.7}$$

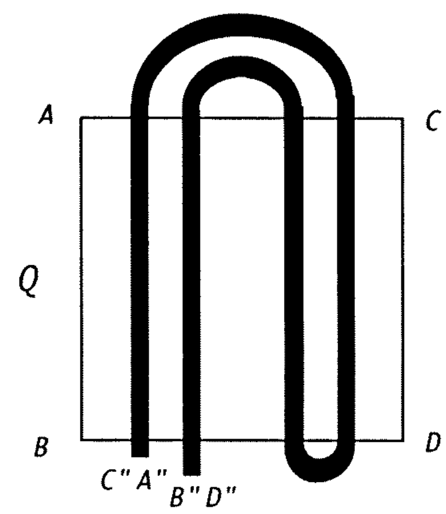
Figure 3.12 shows that the inverse map also has $'Q_0$ and $'Q_1$ oriented the same way as Q, and the well ordered symbols for the past are therefore given by the same algorithm.

The same map but with a reflection around the vertical axis gives $\tilde{Q}_0$ and $\tilde{Q}_1$ oriented as Q while both $'\tilde{Q}_0$ and $'\tilde{Q}_1$ is oriented opposite to Q and the algorithm becomes

$$\begin{array}{llll} \text{if} & t > 0 & \text{then} & w_t = s_t \\ \text{if} & t \leq 0 & \text{then} & w_t = \begin{cases} s_t & \text{if } \ t \ \text{ even} \\ 1 - s_t & \text{if } \ t \ \text{ odd} \end{cases} \end{array} \tag{3.8}$$

Figures 3.13 and 3.14 show a horseshoe map and its inverse map where both intersections change the orientation in future and in past. The well ordered symbols and the symbolic values are obtained by

$$\begin{array}{llll} \text{if} & t > 0 & \text{then} & w_t = \begin{cases} s_t & \text{if } \ t \ \text{ odd} \\ 1 - s_t & \text{if } \ t \ \text{ even} \end{cases} \\ \text{if} & t \leq 0 & \text{then} & w_t = \begin{cases} s_t & \text{if } \ t \ \text{ even} \\ 1 - s_t & \text{if } \ t \ \text{ odd} \end{cases} \end{array} \tag{3.9}$$



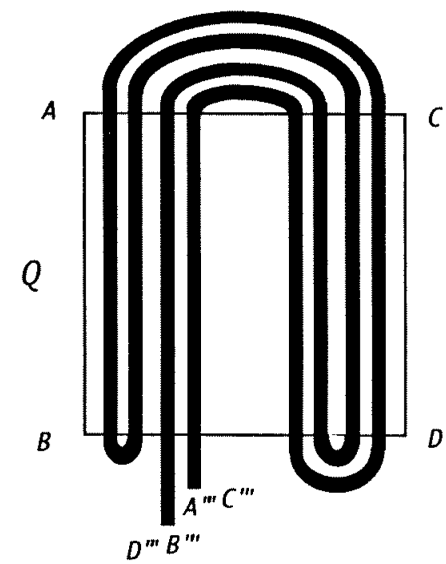


Figure 3.8: The image $\tilde{g}^{(2)}(Q)$ when applying the Smale horseshoe map with reflection twice.

Figure 3.9: The image $\tilde{g}^{(3)}(Q)$ when applying the Smale horseshoe map with reflection three times.

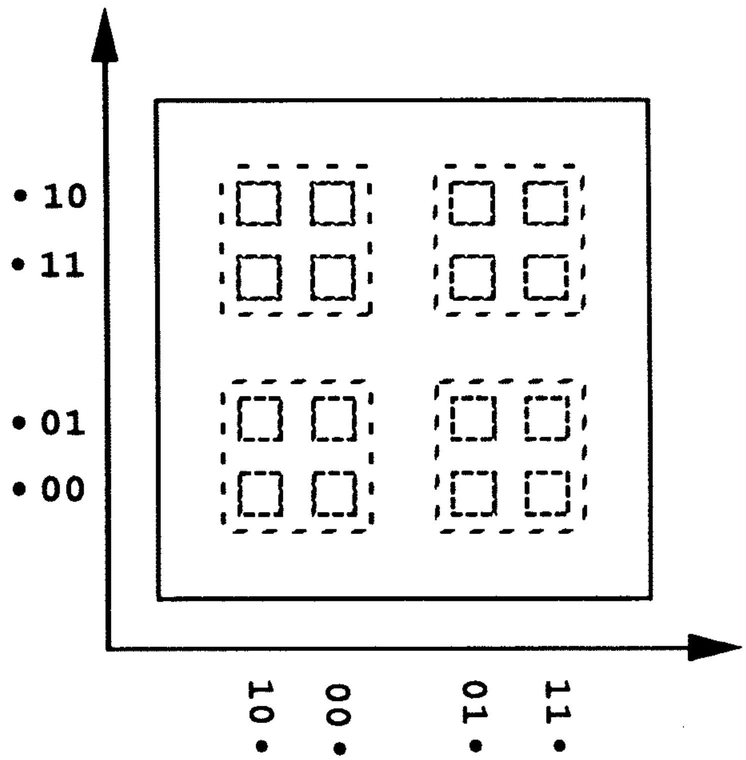


Figure 3.10: The square Q and the sets Λ_1 and Λ_2 of the Smale horseshoe with reflection. and the labels $\cdot s_1 s_2$ and $s_{-1}s_0\cdot$.

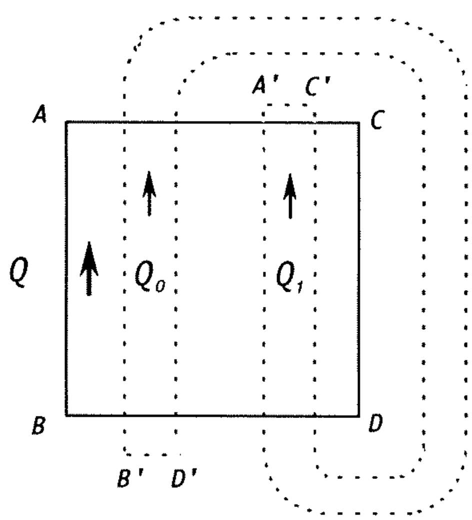


Figure 3.11: The once folding Smale horseshoe map with different folding.

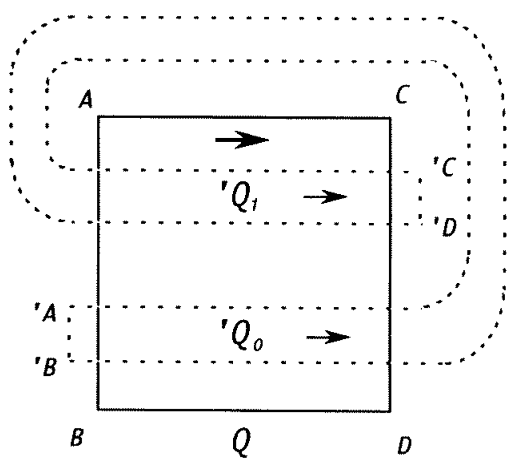


Figure 3.12: The inverse mapping of the Smale horseshoe in figure 3.11.

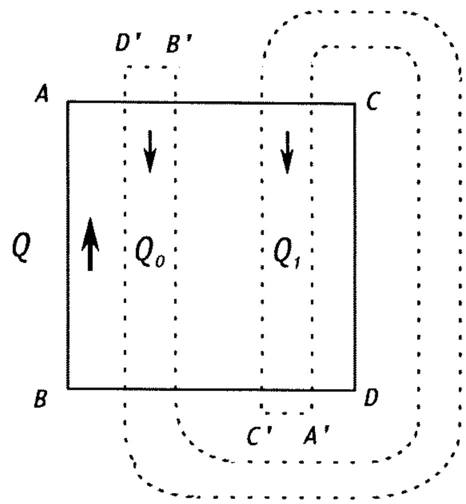


Figure 3.13: A once folding Smale horseshoe map with all symbols reversing the ordering.

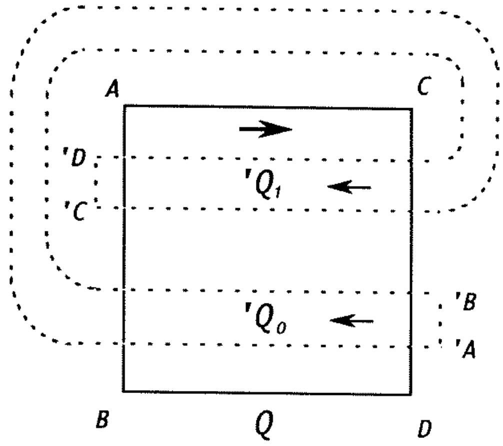


Figure 3.14: The inverse mapping of the Smale horseshoe in figure 3.13.

3.2.2 Twice-folding maps

Figure 3.15 shows a twice-folding horseshoe map with three intersections $g(Q) \cap Q$. The intersections are enumerated in such a way that when $x \in Q$ moves from the bottom to the top, then $g(x)$ visits the intersections in the order: $Q_0 \to Q_1 \to Q_2$. The intersections Q_0 and Q_2 have the same orientation as Q, while Q_1 is oriented opposite to Q. The inverse map is drawn in figure 3.16 and the three intersections $'Q_0$, $'Q_1$, and $'Q_2$ have respectively the same, the opposite, and the same orientation as Q.

The well defined future symbolic dynamics has the same ordering as a bimodal, one-dimensional map with $f'(x) > 0$ for $x < x_{c1}$, and $x > x_{c2}$ and with $f'(x) < 0$ for $x_{c1} < x < x_{c2}$. From the symbols s_t defined as in (3.2) with $k \in \{0, 1, 2\}$ we get

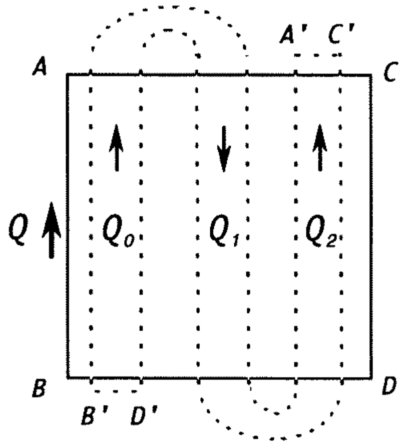


Figure 3.15: The twice folding Smale horseshoe map with simple folding.

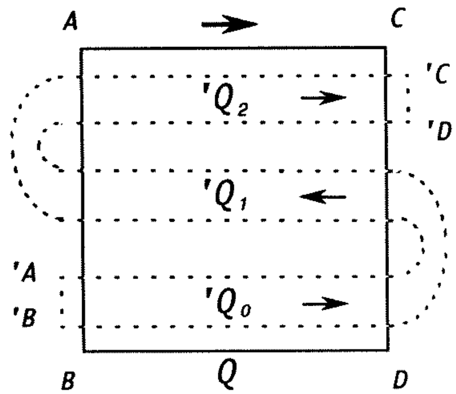


Figure 3.16: The inverse mapping of the Smale horseshoe in figure 3.15.

well ordered symbols w_t from algorithm (2.4) with $n = 2$

$$\begin{aligned}
w_1 &= s_1 \\
p_1 &= \begin{cases} \;\;1 & \text{if} \quad s_1 = 0 \quad \text{or} \quad s_1 = 2 \\ -1 & \text{if} \quad s_1 = 1 \end{cases} \\
w_t &= \begin{cases} s_t & \text{if} \quad p_{t-1} \;=\; \;\;1 \\ 2 - s_t & \text{if} \quad p_{t-1} \;=\; -1 \end{cases} \\
p_t &= \begin{cases} \;\;p_{t-1} & \text{if} \quad s_t = 0 \quad \text{or} \quad s_t = 2 \\ -p_{t-1} & \text{if} \quad s_t = 1 \end{cases} \\
\gamma &= 0.w_1 w_2 w_3 \ldots = \sum_{t=1}^{\infty} \frac{w_t}{3^t}
\end{aligned} \tag{3.10}$$

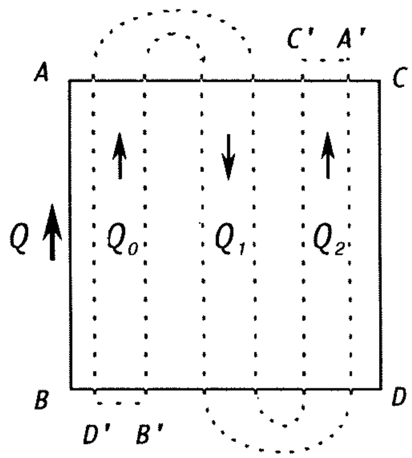


Figure 3.17: The twice folding Smale horseshoe map with reflection.

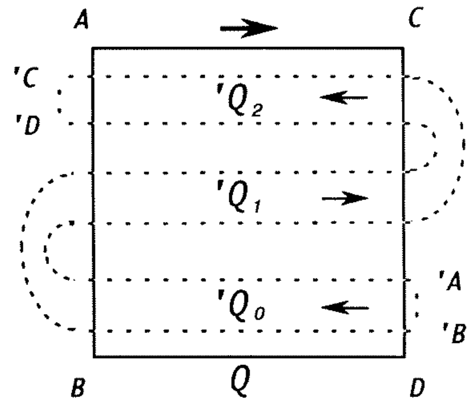


Figure 3.18: The inverse mapping of the Smale horseshoe in figure 3.17.

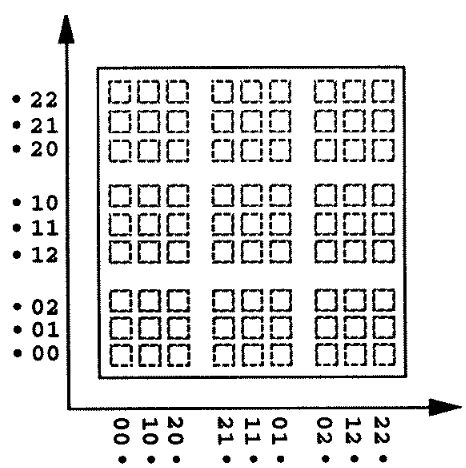


Figure 3.19: The set Λ_2 for the horseshoe in figure 3.15 with symbols $\cdot s_1 s_2$ and $s_{-1} s_0 \cdot$.

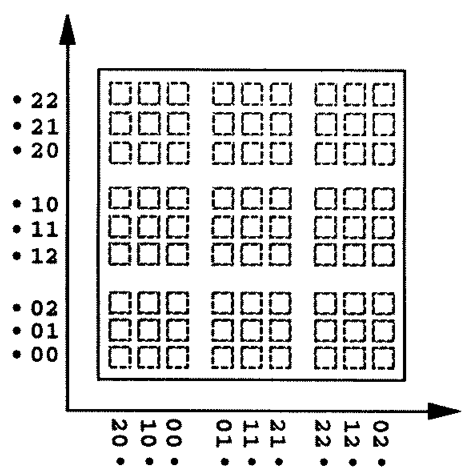


Figure 3.20: The set Λ_2 for the horseshoe in figure 3.17 with symbols $\cdot s_1 s_2$ and $s_{-1} s_0 \cdot$.

The past well ordered symbols are obtained by a similar algorithm:

$$
\begin{aligned}
w_0 &= s_0 \\
p_0 &= \begin{cases} \;\;1 & \text{if} \;\; s_0 = 0 \;\; \text{or} \;\; s_0 = 2 \\ -1 & \text{if} \;\; s_0 = 1 \end{cases} \\
w_t &= \begin{cases} s_t & \text{if} \;\; p_{t+1} = \;\;1 \\ 2 - s_t & \text{if} \;\; p_{t+1} = -1 \end{cases} \\
p_t &= \begin{cases} \;\;p_{t+1} & \text{if} \;\; s_t = 0 \;\; \text{or} \;\; s_t = 2 \\ -p_{t+1} & \text{if} \;\; s_t = 1 \end{cases} \\
\delta &= 0.w_0 w_{-1} w_{-2} \ldots = \sum_{t=0}^{\infty} \frac{w_{-t}}{3^{(t+1)}}
\end{aligned}
\tag{3.11}
$$

We can also add to this two fold map a reflection. Applying a reflection around the vertical axis before stretching and folding gives the figure 3.17, and the inverse mapping in figure 3.18. The well ordered future symbols are the same for the vertically reflected map as for the not reflected map. The well ordered past symbols are obtained by the algorithm

$$
\begin{aligned}
w_0 &= s_0 \\
p_0 &= \begin{cases} \;\;1 & \text{if} \;\; s_0 = 0 \;\; \text{or} \;\; s_0 = 2 \\ -1 & \text{if} \;\; s_0 = 1 \end{cases} \\
w_t &= \begin{cases} s_t & \text{if} \;\; p_{t+1} = \;\;1 \\ 2 - s_t & \text{if} \;\; p_{t+1} = -1 \end{cases} \\
p_t &= \begin{cases} -p_{t+1} & \text{if} \;\; s_t = 0 \;\; \text{or} \;\; s_t = 2 \\ \;\;p_{t+1} & \text{if} \;\; s_t = 1 \end{cases} \\
\delta &= 0.w_0 w_{-1} w_{-2} \ldots = \sum_{t=0}^{\infty} \frac{w_{-t}}{3^{(t+1)}}
\end{aligned}
\tag{3.12}
$$

The sets $\Lambda_2 = g^{(-2)}(Q) \cap g^{(2)}(Q)$ for the two twice-folding maps are drawn in figure 3.19 and figure 3.20 together with the symbols $s_{-1}s_0\cdot$ and $\cdot s_1 s_2$ labeling the 3^4 rectangles.

A more complicated twice folding map is shown in figure 3.21 and its inverse map in figure 3.22. The labeling Q_k is done as before, so that when x moves from the bottom to the top, $g(x)$ visits Q_k in the order $Q_0 \to Q_1 \to Q_2$.

The future intersections Q_k are oriented as in the map above; Q_0 and Q_2 oriented as Q, while Q_1 is oriented opposite to Q. The well ordered future symbols

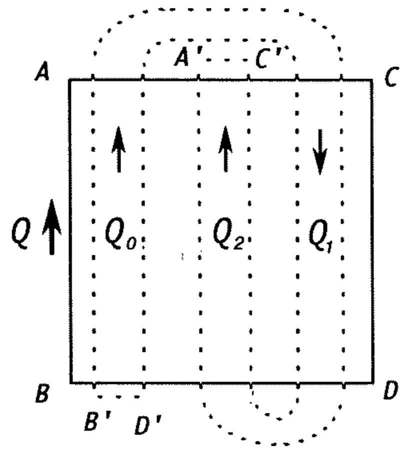


Figure 3.21: The twice folding Smale horseshoe map with more complicated folding.

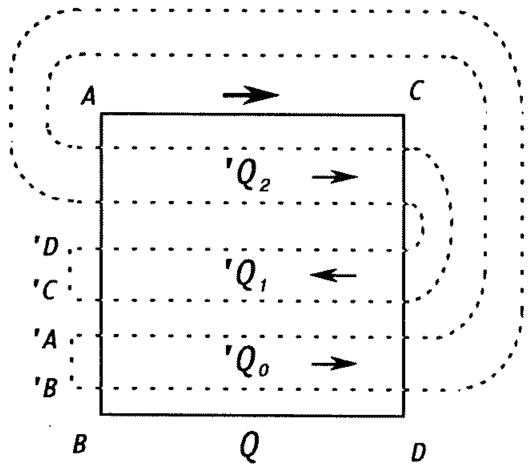


Figure 3.22: The inverse mapping of the Smale horseshoe in figure 3.21.

are therefore obtained by the algorithm (3.10). The intersections $g^{-1}(Q) \cap Q$ are oriented with $'Q_0$ and $'Q_1$ as Q while $'Q_2$ are oriented opposite to Q. This gives the following algorithm for well ordered past symbols:

$$
\begin{aligned}
w_0 &= \begin{cases} 0 & \text{if} \quad s_0 = 0 \\ 1 & \text{if} \quad s_0 = 2 \\ 2 & \text{if} \quad s_0 = 1 \end{cases} \\
p_0 &= \begin{cases} 1 & \text{if} \quad s_0 = 0 \quad \text{or} \quad s_0 = 2 \\ -1 & \text{if} \quad s_0 = 1 \end{cases} \\
\\
w'_t &= \begin{cases} 0 & \text{if} \quad s_t = 0 \\ 1 & \text{if} \quad s_t = 2 \\ 2 & \text{if} \quad s_t = 1 \end{cases} \\
w_t &= \begin{cases} w'_t & \text{if} \quad p_{t+1} = 1 \\ 2 - w'_t & \text{if} \quad p_{t+1} = -1 \end{cases} \\
p_t &= \begin{cases} p_{t+1} & \text{if} \quad s_t = 0 \quad \text{or} \quad s_t = 2 \\ -p_{t+1} & \text{if} \quad s_t = 1 \end{cases} \\
\\
\delta &= 0.w_0 w_{-1} w_{-2} \ldots = \sum_{t=0}^{\infty} \frac{w_t}{3^{(t+1)}}
\end{aligned}
\tag{3.13}
$$

The well ordered symbols can be worked out for any horseshoe map like these by observing weather the intersections are oriented as or opposite to Q and flipping the symbols if the symbols correspond to oppositely oriented rectangle.

One example we will use later is an n-folding map with all future intersections $Q_0, Q_1, \ldots, Q_n$ oriented opposite to Q. This is the generalized folding of the once-folding map in figure 3.13. The algorithm giving the well ordered symbols is

$$
\begin{aligned}
&\text{if} \quad t > 0 \quad \text{then} \quad w_t = \begin{cases} s_t & \text{if} \quad t \quad \text{odd} \\ n - s_t & \text{if} \quad t \quad \text{even} \end{cases} \\
&\text{if} \quad t \leq 0 \quad \text{then} \quad w_t = \begin{cases} s_t & \text{if} \quad t \quad \text{even} \\ n - s_t & \text{if} \quad t \quad \text{odd} \end{cases}
\end{aligned}
\tag{3.14}
$$