Part IV

Hamiltonian systems

Chapter 10

Smooth Hamiltonian systems

In this chapter we will discuss the difficult problem of analyzing a smooth Hamiltonian system. It is not proven for any nontrivial example that there exist a partition of the non-wandering set, and there is no theory analog to the MSS theory which explains the ordering of different bifurcations in these systems. We will show some examples where we can understand bifurcations better than before by using the results we have obtained on folding maps and on billiard systems. These examples support the conjectures and speculations we present below concerning the smooth Hamiltonian systems and we suggest future investigations of these problems.

In an area preserving smooth system, e.g. a Hamiltonian system, there is a non-wandering set of stable and unstable orbits. The major difference from the smooth dissipative system is that the stable orbits are not attracting neighboring orbits but are surrounded by KAM tori [134]. The determinant of the Jacobian of a dissipative system is less than 1 while for the area preserving map the determinant is 1.

From a two-dimensional Hamiltonian we can find the Jacobian for the canonical variables (I, θ) [134] (also called the monodromy matrix) and since the determinant is 1 a period n orbit is hyperbolic (unstable) if

$$|\mathrm{Tr}\,\mathbb{J}| > 2 \tag{10.1}$$

and the orbit is elliptic (stable) if

$$|\mathrm{Tr}\,\mathbb{J}| < 2 \tag{10.2}$$

If $|\mathrm{Tr}\,\mathbb{J}| = 2$ we have a bifurcation point. If the periodic orbit is elliptic then the eigenvalues are

$$\lambda_{1,2} = e^{\pm i\sigma} \tag{10.3}$$

$$\sigma = \arccos\left(\frac{\mathrm{Tr}\,\mathbb{J}}{2}\right) \tag{10.4}$$

and there is a bifurcation of a periodic orbit for all rational values of $\sigma/2\pi$ [148, 8, 94]. The typical bifurcation for $\sigma/2\pi = p/q$ is the creation of two period $q \cdot n$ orbits in the Poincaré map where one orbit is elliptic and one is hyperbolic and the orbits have q points in a chain surrounding each point of the period n orbit in the Poincaré plane. If $q \leq 5$ the orbits may have a different bifurcation depending on the system. The general classification of bifurcations is given by Meyer [148] and is used to show general scaling behavior of bifurcations [94].

This general theory does not predict neither the shape of the bifurcated orbit in the configuration space nor how many times a given periodic orbit has the same value of σ scanning a parameter line.

10.1 Hamiltonian Hénon maps

The once folding maps are for some parameter values area preserving maps and could in principle be constructed as a Poincaré map of a Hamiltonian flow. The Hénon map has det $\mathbb{J} = -b$ and is area preserving for $|b| = 1$ and also the Lozi map is area preserving for $|b| = 1$.

In chapter 4 we found that some cusp bifurcations have to be exactly at $b = 1$ or $b = -1$ because of symmetries in the symbol strings describing the orbits. In a 2^n-dimensional symbolic parameter space the area preserving maps seem not to correspond to a simple line or surface but to points characterized by a special symmetry in the strings giving the values $\kappa_0, \kappa_1, \kappa_{10}, \kappa_{00}, \kappa_{01}, \kappa_{11}, \ldots$. Similarly in the pruning front language it is not obvious which pruning fronts that give an area preserving map, but the points (δ, γ) on the pruning front must have special symmetries in the symbol strings.

That we at all can describe bifurcations e.g. the period 4 cusp $\{\overline{1011}, \overline{1000}, \overline{1001}\}$ in the symbolic parameter space $(\kappa_{10}, \kappa_{00}, \kappa_{01}, \kappa_{11})$ indicates that the non-wandering set at $|b| = 1$ can be described by the binary symbolic dynamics. However conjecture 1 which gave a continuous partition curve of the Hénon map in the (x_t, x_{t+1}) space assumed that the unstable manifold is not dense in (x_t, x_{t+1}) which is not necessarily true for $|b| = 1$. When there is a bifurcation of a turning points the partition curve may break up into not connected parts. A not connected partition curve may be acceptable if it goes through the unstable manifold in a unique way, but the partition of the area conserving Hénon map is not understood and further investigations is needed.

10.2 The $(x^2y^2)^{1/a}$ potential

One group of smooth Hamiltonians which seem to be promising candidates for a good symbolic description is the two-dimensional potentials $V(x,y)$ that have a shape which can be compared with a billiard system. We can consider these systems to be created by starting with a billiard potential which is 0 in the domain accessible to the particle and $+\infty$ in the forbidden regions and then smoothen this hard potential to a soft potential.

One example of this is the potential

$$V(x,y) = \left(x^2y^2\right)^{\frac{1}{a}} \tag{10.5}$$

which was investigated by Dahlqvist and Russberg [57, 58] and other authors. In the limit $a \to 0$ this potential becomes the hyperbola billiard (7.10) which we in section 7.4 found was described by a slightly pruned trinary alphabet. Dahlqvist and Russberg adiabatically followed periodic orbits starting at the hyperbola billiard $a = 0$ and letting the parameter a increase slowly. They found that an unstable periodic orbit in the hyperbola billiard smoothly changes and is hyperbolic when a increases, and some orbits have a parameter interval (a window) where the orbit is elliptic, and then the orbit disappears in a bifurcation and does not exist for larger values of a. Other orbits bifurcate and disappear without becoming stable. They described a number of bifurcations and found that the symbolic description of the orbits merging into each other is similar. They conjectured [58] that all orbits pruned during one bifurcation cascade, have a symbolic description consisting of combinations of two symbolic strings.

Figure 10.1 shows Dahlqvist and Russbergs calculation of $\mathrm{Tr}\,\mathbb{J}$ as a function of a for some periodic orbits. The symbols they use are slightly different than those used here. In figure 10.2 the same bifurcations are drawn with dashed lines indicating hyperbolic orbits and solid lines indicating elliptic orbits. For clarity the parameter axis is not at correct scale and the vertical axis is a sketch of a possible point in a Poincaré map.

Dahlqvist and Russberg showed that the orbit $\overline{S} = \overline{1(32)^4 34(23)^4 2}$ is stable in a window that includes the parameter value $a = 1$, and therefore that an earlier conjecture claiming that the potential $V = x^2y^2$ is completely chaotic was wrong [57].

We are now interested in the symbolic description of the orbits bifurcating in one bifurcation scenario. Dahlqvist and Russbergs conjecture is that all orbits in the cascade in figure 10.2 a) are described by the symbol strings of the form

$$\ldots 2t_{-1}434t_0232t_1434t_223\ldots \tag{10.6}$$

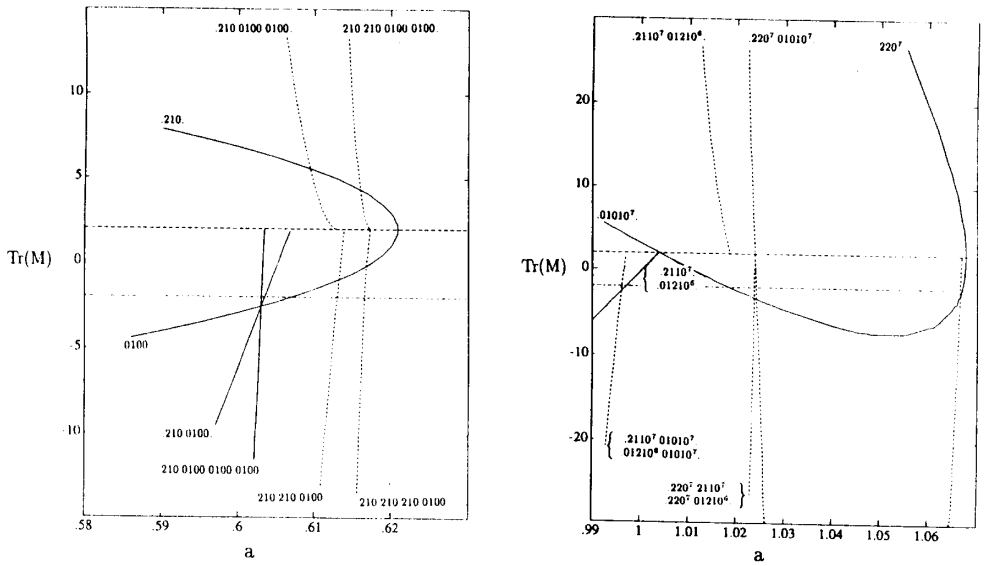


Figure 10.1: The $\mathrm{Tr}\,\mathbb{J}$ as a function of a for some periodic orbits. From [58].

with t_i either symbol 3 or no symbol, and that the orbits in figure 10.2 b) are described by the strings

$$\ldots 1s_{-1}(32)^4t_{-1}4t_0(23)^4s_0 1s_1(32)^4t_1 4t_2(23)^4s_2\ldots \tag{10.7}$$

with s_i either 2 or no symbol, and t_i either symbol 3 or no symbol.

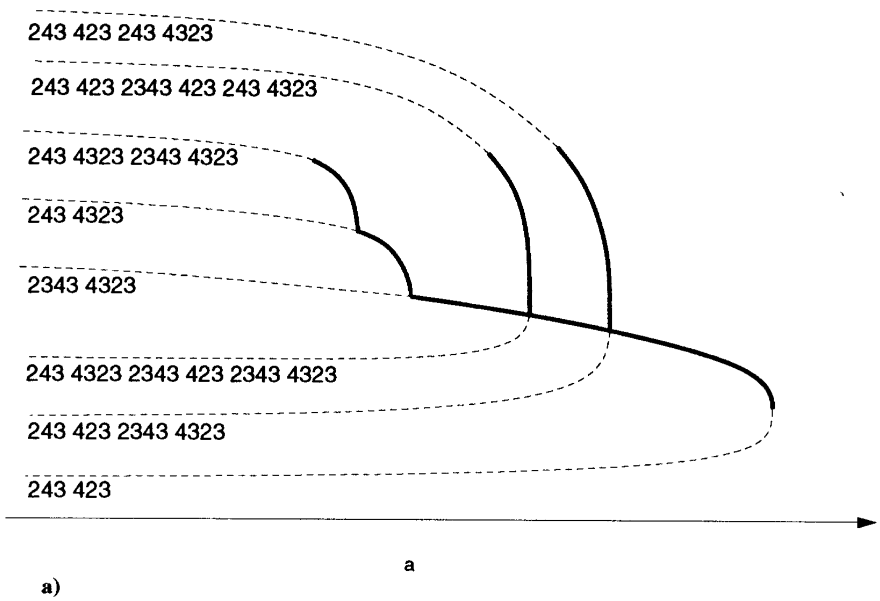
We now compare these bifurcations to the bifurcation families for the billiard systems in chapter 9 and we find that the strings describing a billiard family of orbits are the same strings that give the orbits bifurcating in the bifurcation trees of the potential (10.5). The 4 disk system with $r_c = 2.104231\ldots$, figure 10.4, is the bifurcation point of the orbits with symbolic description (10.6) and for $r_c = 2.0312\ldots$ there is a singular bifurcation of the orbits (10.7), figures 9.5 and 9.6.

We now state a general conjecture of bifurcations of "billiard like" smooth potentials.

Conjecture 2 *In a billiard like smooth Hamiltonian system the orbits bifurcating in one bifurcation cascade are the orbits of one family bifurcating at a singular parameter r_c for a corresponding billiard system.*

We can also state an other interesting conjecture.

Conjecture 3 *The symbolic description of the bifurcation family of the billiard system predicts the minimum number of times a periodic orbit has to bifurcate with the same complex eigenvalues $\lambda_{1,2} = e^{\pm i\sigma}$ for any one-dimensional parameter path from the complete horseshoe repellor to the parameter value where the orbit disappear.*



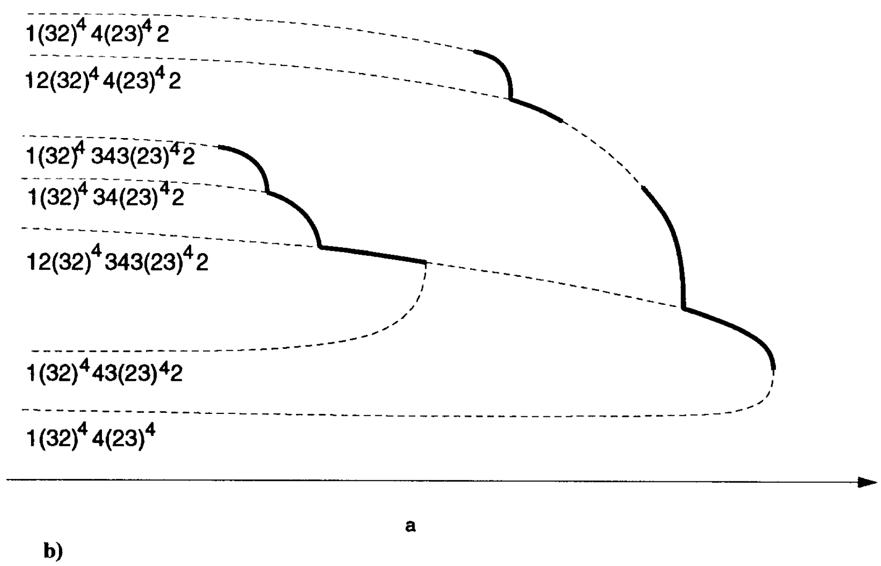


Figure 10.2: A sketch of the bifurcation trees of the orbits in figure 10.1. Solid lines are elliptic orbits while dashed lines are hyperbolic orbits.

One major difficulty in these conjectures is that we have not precisely defined a "billiard like" Hamiltonian and which billiard one associates with a specific potential. A guiding line may be that a smooth slope is represented as a hard billiard wall, convex and concave walls in the billiard are similarly shaped in the smooth potential and the symmetries in configuration space have to be the same.

The potential (10.7) has four hills which we identify with the 4 disks in the 4-disk system. We could have studied the hyperbola billiard (7.10) and changed the hard hyperbola walls to find the bifurcations, but the bifurcation families would be the same in these two billiards and the 4-disk system is more convenient. In some cases the 4-disk system may give problems because it has the corner pruning which is not present in the potential (10.7). The singular bifurcation of a billiard is when the orbit is tangent to the disk. In the smooth potential there is no sharp wall to be tangent to, but in Dahlqvist and Russbergs plot of orbits in the configuration space, figure 10.3, we find that orbits that are going to bifurcate together, move closer such that the curves corresponding to the tangent line in figure 10.4 straighten out and move closer to each other. We then get a kind of tangent curve off a soft wall. This kind of curve is what we have to identify as a singular orbit or a turning point in the smooth potential. The orbit is tangential to a constant energy line but the position is not given as simple as for the billiard system. To find this orbit in general is an open question.

We can illustrate the conjectures 2 and 3 by the two examples in figure 10.2. In figure 10.2 a) the orbit $\overline{2343\,4323}$ becomes elliptic in a bifurcation where the orbit $\overline{243\,4323}$ disappears and then the elliptic orbit changes once through each winding numbers σ and then the orbit disappears together with the unstable orbit $\overline{243\,423}$. The bifurcation where the elliptic orbit becomes hyperbolic is a symmetry breaking bifurcation which in some Poincaré map is a bifurcation from a fixed point to a period two orbit. In the 4-disk system all orbits in the family (10.6) bifurcate for $r_c = 2.104231\ldots$ where one line is tangent to a disk, figure 10.4.

The family $\ldots 1s_0(32)^4t_04t_1(23)^4s_1\ldots$ in the 4-disk system, figure 9.5, has one more tangent line do the disks than the $\ldots 2t_0434t_123\ldots$ family. If the symmetric orbit $\overline{12(32)^4343(23)^42}$ bifurcates in a similar way as the orbit $\overline{2343\,4323}$ then $\overline{12(32)^4343(23)^42}$ has to bifurcate two times to bifurcate together with the different symmetry broken orbits. For these two examples we find that the number of tangent lines of the billiard orbit at r_c is equal to the number of stable windows of the orbit. We may conjecture that this is a general feature such that the number of tangent lines in the singular bifurcation gives the number of times the short orbit has to go through a stable window.

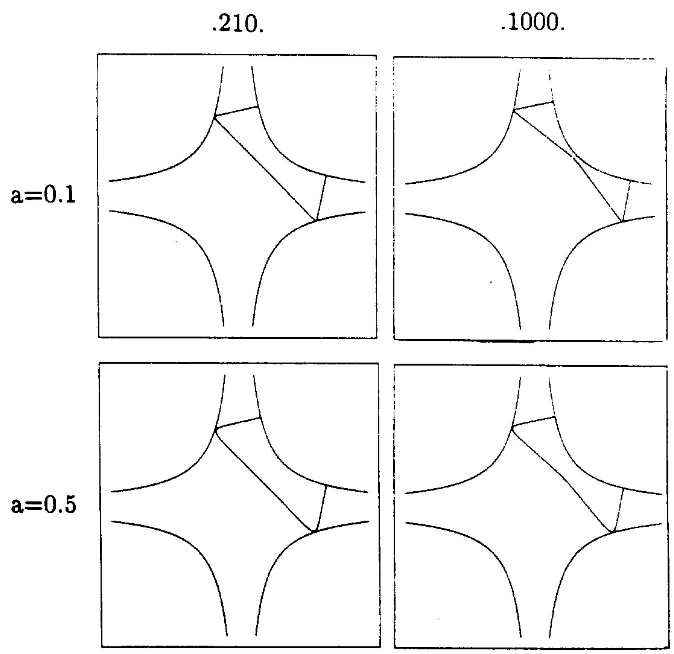


Figure 10.3: The orbits $\overline{243\,423}$ and $\overline{2343\,4323}$ in potential (10.7) for two different parameter values. From [58].

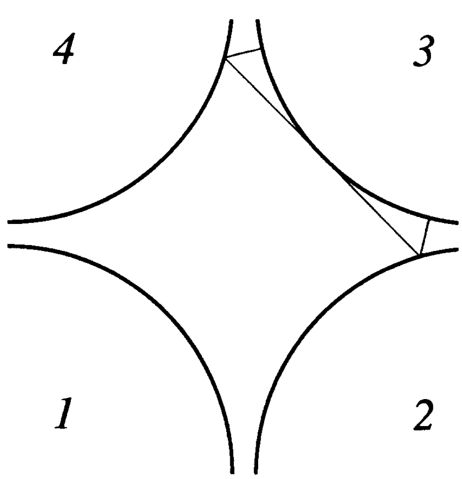


Figure 10.4: The orbit $\overline{243\,423}$ in the 4 disk system for $r_c = 2.104231\ldots$.

In figure 10.2 a) we also find that orbits bifurcating in the stable window at rational values of $\sigma/2\pi$ are described by the symbol string (10.6). The length of the symbol string gives approximately the length of the orbit in the potential.

The non-periodic symbol strings from (10.6) and (10.7) may be orbits from the accumulation point of n-tuplings of stable periodic orbits or from quasiperiodic orbits created between the creation of periodic orbits at irrational values of $\sigma/2\pi$

In the two examples shown here the shortest critical billiard orbit is tangent to the wall 2 or 4 times. If the orbit is tangent only one time we expect the bifurcation where the shortest elliptic orbit turns hyperbolic to be a period doubling bifurcation instead of the symmetry breaking bifurcation.

10.3 Parabola shaped potentials

We will sketch here how an investigation of bifurcations in a smooth potential can be interpreted in terms of symbolic dynamics of a corresponding billiard.

10.3.1 NELSON

Baranger and Davies [23] have carefully studied bifurcations in a potential

$$V(x,y) = \left(y - \frac{x^2}{2}\right)^2 + \frac{x^2}{20} \tag{10.8}$$

which they called NELSON, and other more complicated potentials [1]. They studied the bifurcations of periodic orbits as a function of the energy for this potential. Figure 10.5 shows equipotential contours of potential (10.8). This system without any other parameters does not have a limit of a hard billiard system and Baranger and Davies did not try to give a symbolic description of orbits but denoted them different names according to how they look and how they bifurcate. The potential is just chosen as one typical system of a parabolic shape and it is no reason to expect that the increasing energy , E, should be a more special parameter path than a change of a parameter in the potential as in the $(x^2y^2)^{1/a}$ potential.

We will now try to give the periodic orbits as a symbolic description by replacing the soft walls with hard billiard walls and find singular bifurcations which corresponds to a bifurcation tree in the smooth potential. In addition to the billiard bifurcation where the singular orbit is tangent to a dispersing wall we also find it necessary to allow two consecutive bounces in a focusing wall to merge into one bounce.

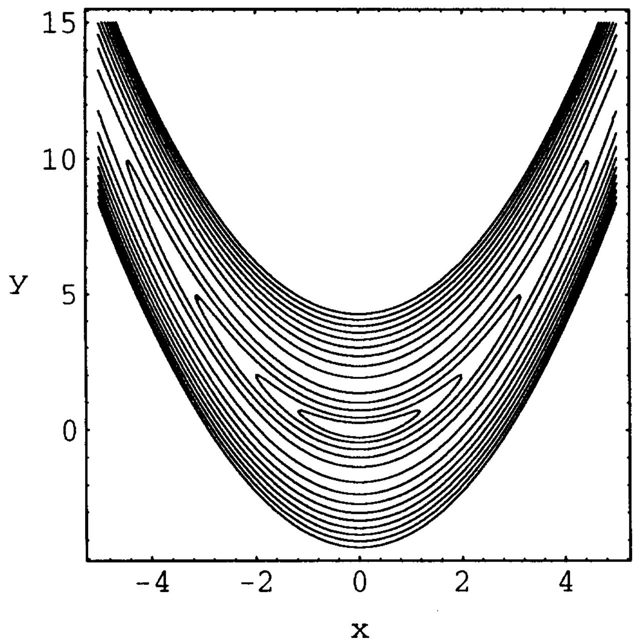


Figure 10.5: Equipotential contours of the potential (10.8).

Baranger and Davies have chosen to plot the bifurcations in the (E, τ) plane where E is the energy and τ is the length of the periodic orbit. Some of their bifurcation diagrams are given in figure 10.6. In these diagrams orbits are denoted by different letters and the plots of some of the periodic orbits in (x, y) are given in figure 10.7.

We will define a billiard system with a symbolic dynamics and identify the orbits in figure 10.7 with a symbol string. A "smiling billiard" is drawn in figure 10.8. This is just a hand-drawn billiard which is not ergodic and the periodic orbit is also just sketched by hand. A change of parameter is a change in the shape of the walls. We define the symbols:
$s = 1$ for a bounce off the dispersing wall,
$s = 2$ for a counterclockwise bounce off the focusing wall,
$s = 3$ for a clockwise bounce off the focusing wall and
$s = 4$ for a normal bounce off the focusing wall.
If the normal bounce $s = 4$ is on the symmetry line $x = 0$ we denote it $4'$. We assume this is a covering (and heavily pruned) alphabet. A number of periodic orbits of the billiard are drawn in figures 10.8 and 10.9 together with the symbol string $\overline{S}$ and the Baranger–Davies name for the corresponding smooth orbit.

In figure 10.6 we find that when the smooth orbit **A**; $\overline{133122}$ changes from elliptic to hyperbolic then the orbit **q2**; $\overline{13131212}$ is born. In our smiling billiard there exists

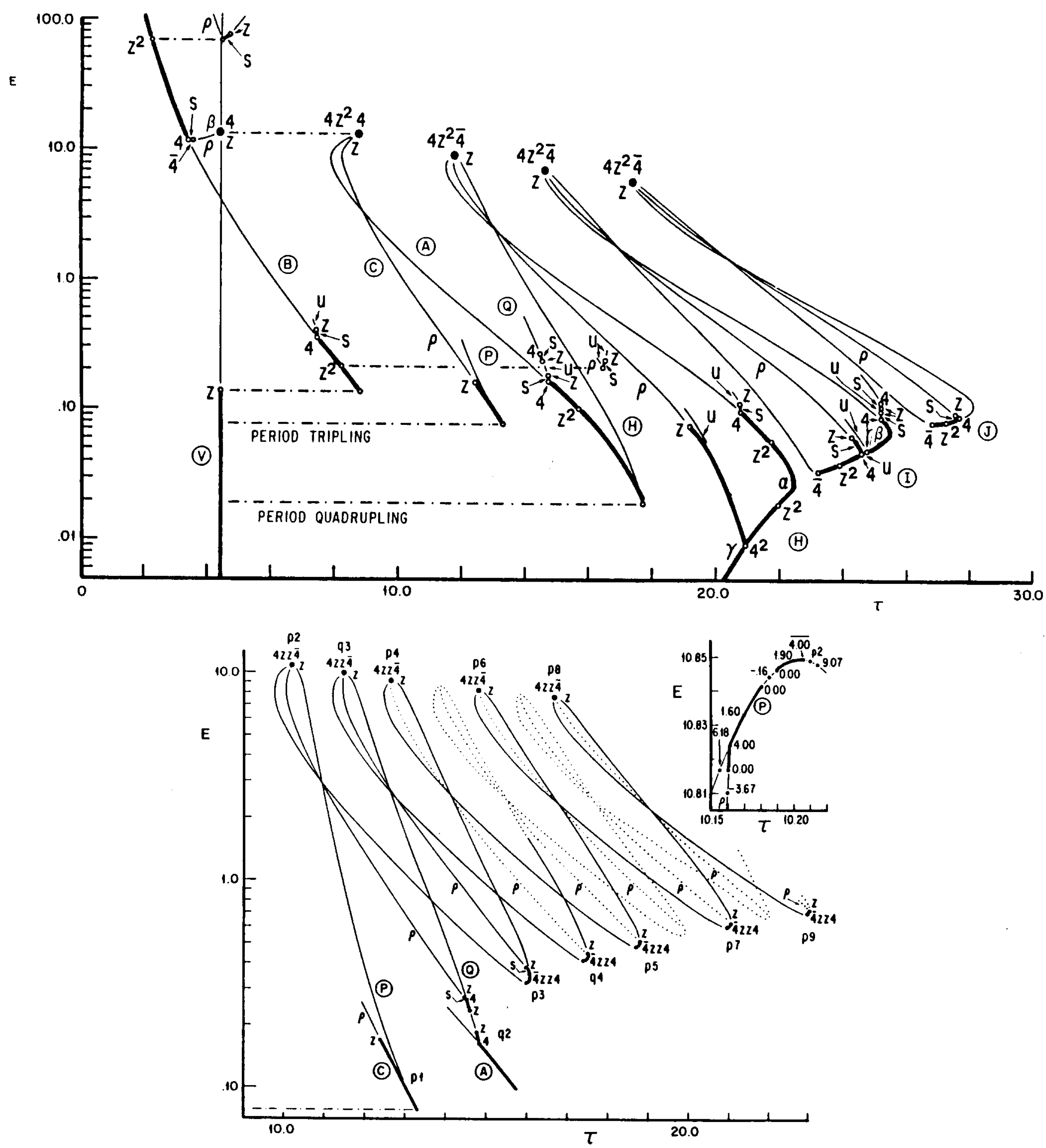


Figure 10.6: The bifurcation diagram (E, τ) of the potential (10.8). From [23].

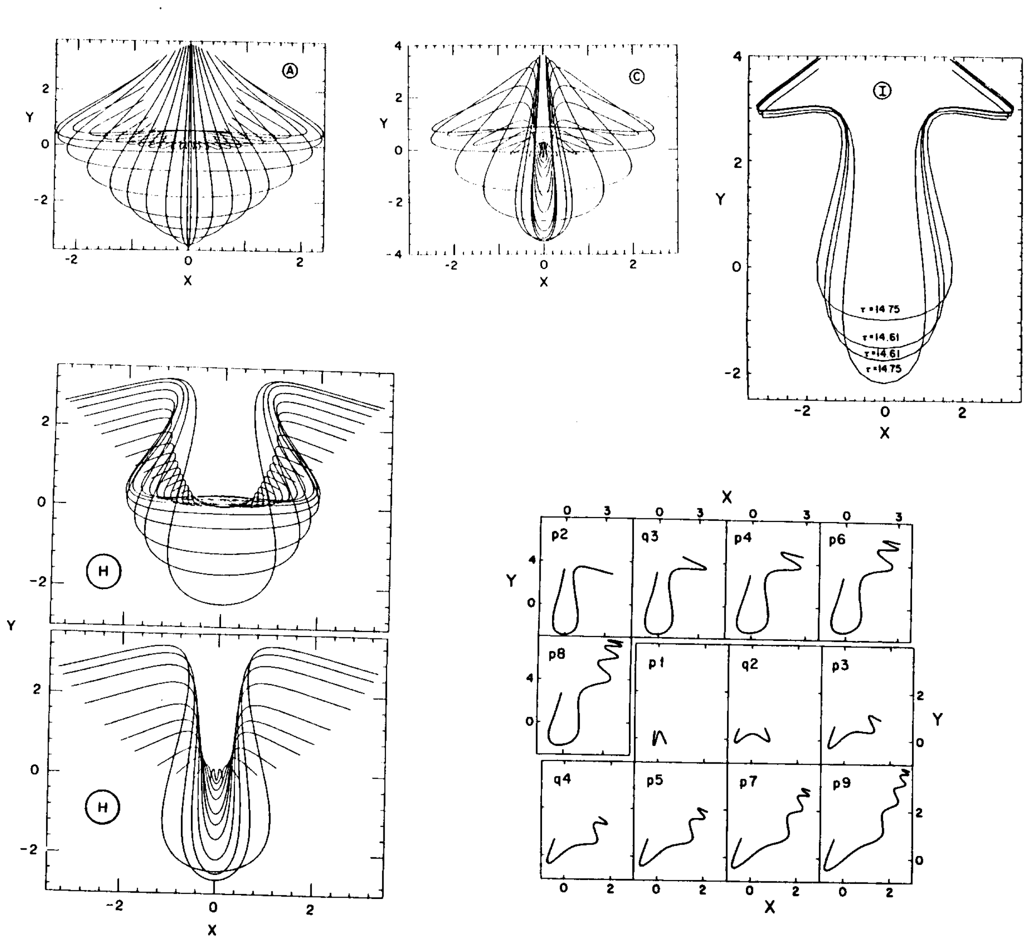


Figure 10.7: Some periodic orbits in the potential (10.8). From [23].

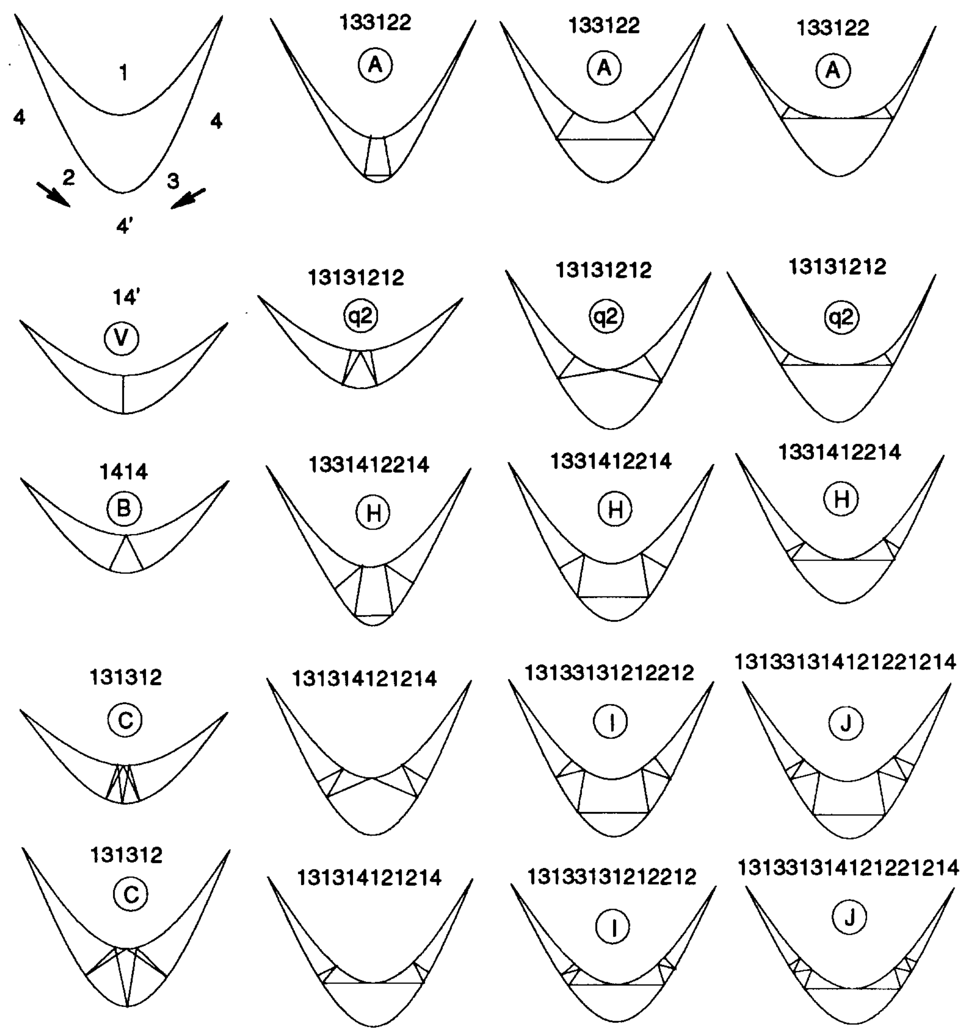


Figure 10.8: The symbols and some periodic orbits in the "smiling billiard".

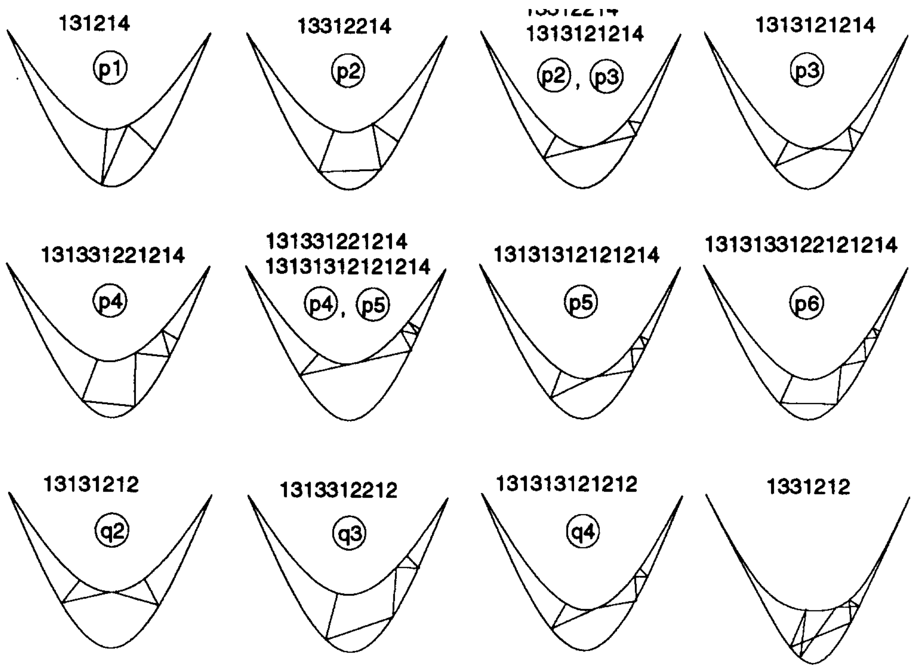


Figure 10.9: Periodic orbits in the "smiling billiard".

a singular bifurcation of these two orbits where the orbits have a line tangential to the dispersive wall. This orbit is drawn in figure 10.8. The singular bifurcation gives a change of symbols $33 \leftrightarrow 313$ or $22 \leftrightarrow 212$. The singular bifurcation family of orbits have the symbolic description

$$\ldots 13t_{-1}312t_0213t_1312t_22\ldots \tag{10.9}$$

with t_i either 1 or no symbol. One other periodic orbit in this bifurcation family is $\overline{1331212}$ which is drawn in figure figure 10.9 and which we identify with an orbit bifurcating from **q2**; $\overline{13131212}$ in figure 10.6.

An other example of a tangent type of a singular orbit in the billiards is the family

$$\ldots 1313t_{-1}312t_021413t_1312t_2214\ldots \tag{10.10}$$

with t_i either 1 or no symbol. In figure 10.9 the singular orbit is drawn and the two orbits **p2**; $\overline{13312214}$ and **p3**; $\overline{1313121214}$ are drawn for a different parameter values. The orbits **p2** and **p3** bifurcate together in the (E, τ) plane in figure 10.6.

It seems to be necessary to make an ad. hoc. assumption of a second bifurcation in the smiling billiard. We allow one bounce in the focusing wall close to $x = 0$ to split into two bounces. This bifurcation implies the change of symbols $3 \leftrightarrow 33$ or $2 \leftrightarrow 22$. This is not an bifurcation that are a singular bifurcation in a billiard but in the region below origin in the smooth potential a path changes very smoothly and a billiard model may not be a good approximation here. An example of a family from this bifurcation is

$$\ldots 13r_0 12s_0 1413r_1 12s_1 14\ldots \tag{10.11}$$

with r_i either 3 or no symbol and with s_i either 2 or no symbol. The symbol 4 can turn into 2 or 3 if the combination of r_i and s_i gives an orbit that is not symmetric. The two orbits **p1**; $\overline{131214}$ and **p2**; $\overline{13312214}$ in figure 10.6 are members of this family. A periodic orbit may be member of both these two kind of singular bifurcations and this gives the zig-zag structure in figure 10.6 b) which Baranger and Davies calls a "duet of asymmetric librations". The orbit $\overline{1331212}$ in figure 10.9 is a member of the family (10.9) and of the family (10.11) and is the hard version of the smooth orbit Baranger and Davies calls the "first rotational bridge". Figure 10.10 a) shows Baranger and Davies' rotational bridge for different parameter values connecting the orbits **q2** and **p2**. Figure 10.10 b) is the billiard orbit $\overline{1331212}$ connecting the orbits $\overline{13131212}$ and $\overline{13312214}$. The symbolic description of all bifurcation families in this zig-zag structure is

$$\begin{array}{c} \ldots(13)^{k+1}t_0 312t_1 2(12)^k\ldots \\ \ldots(13)^{k+1}t_0 312t_1 2(12)^{k-1}14\ldots \\ \ldots(13)^{k+1}r_0 12s_0(12)^k\ldots \\ \ldots(13)^{k+1}r_0 12s_0(12)^k 14\ldots \end{array} \tag{10.12}$$

with $t_i \in \{1,\emptyset\}$, $s_i \in \{2,\emptyset\}$ and $r_i \in \{3,\emptyset\}$.

It seems possible to describe all bifurcations in this system in terms of a symbolic alphabet but more investigations of the bifurcations in this and similar systems should be done to test this conjecture. The family of orbits obtained by splitting one bounce in the focusing wall into two bounces is made ad. hoc. to fit the description of the smooth system and are not directly motivated from the billiard suggested as a hard model of the potential. From a symbolic dynamics point of view is this a simple creation of symbol strings but from a physical point of view we would like to have a better billiard model that also had this family of orbit as a singular bifurcation.

10.3.2 Størmers problem

A physical problem which has a kind of parabola shaped potential is the motion of a charged particle in a magnetic dipole field. Pioneering numerical investigations of this problem was started by Carl Størmer 1903 [187] who probably was the first to numerically calculate complicated chaotic orbits [122, 123].

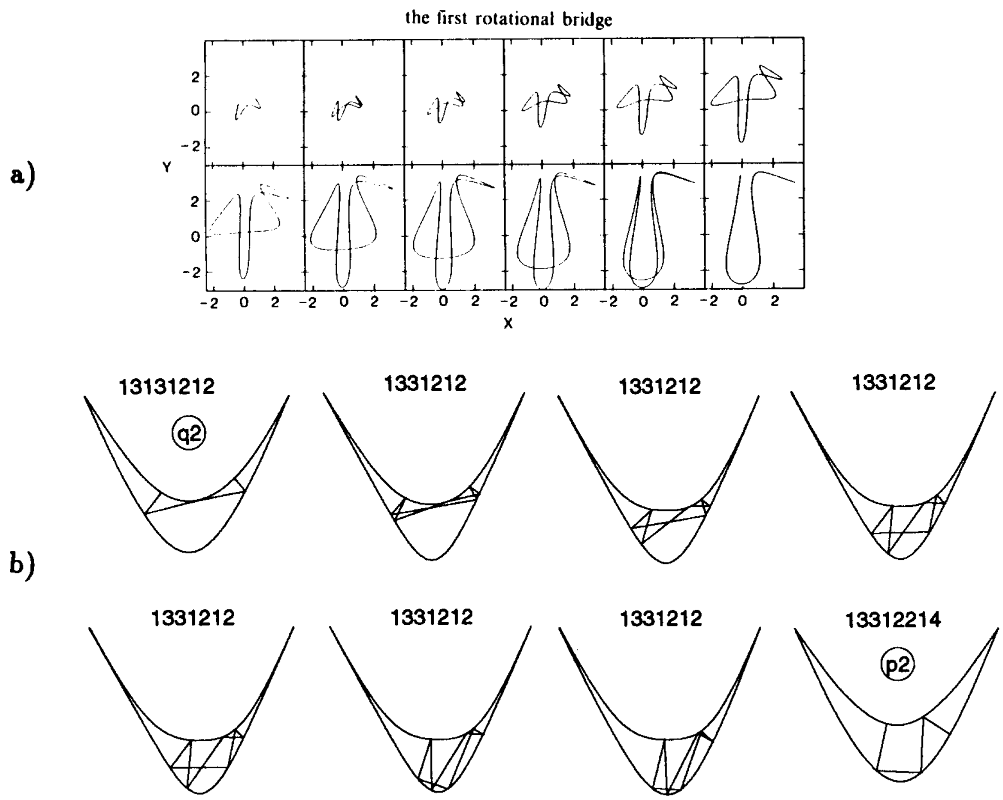


Figure 10.10: a) The "first rotational bridge" of the potential (10.8). From [23]. b) The orbit $\overline{1331212}$ in the "smiling billiard".

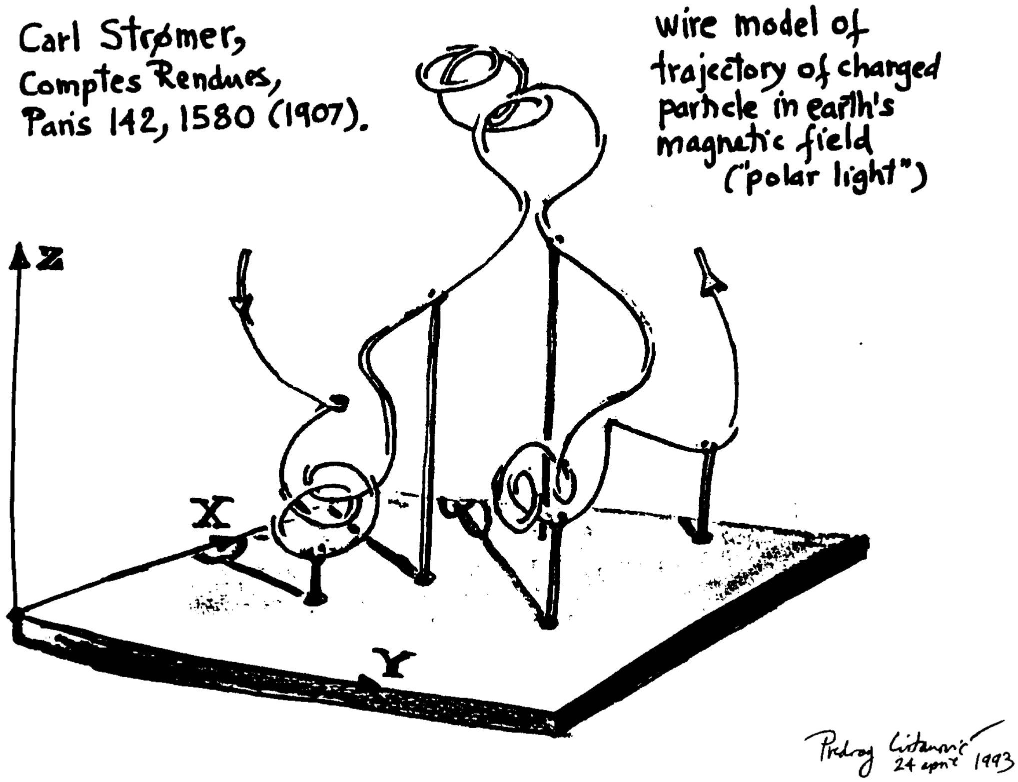


Figure 10.11: The path of a particle in a magnetic dipole field. Drawing by P. Cvitanović after C. Stormer [187]

The problem can be reduced to a two dimensional problem with the Hamiltonian

$$H = \frac{1}{2}(p_\rho^2 + p_z^2) + \frac{1}{2}\left(\frac{1}{\rho} - \frac{\rho}{r^3}\right)^2 \tag{10.13}$$

where ρ, ϕ, z is the cyclical coordinates around the magnetic dipole and r is the distance from the particle to the origin. For low energies the potential is near integrable with KAM tory and it has a shape similar to a parabola but with one path going into the dipole. For slightly larger values of the energy the problem is a scattering problem with a structure comparable to the three-disk problem. For large enough values of the energy the system is a complicated but not chaotic scatter. This system is investigated in several works [31, 38, 62, 64, 123, 124] but a careful study of the bifurcation structure has not been done. We expect that we can describe the bifurcation diagrams for the Størmer problem by a symbolic dynamics obtained from a corresponding billiard system as we could do for the potential (10.8). Figure 10.11 shows one path of a scattered particle in the dipole field.