what is ‘chaos’?
a field theorist stroll through Bernoullistan

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ChaosBook.org/overheads/spatiotemporal
$\rightarrow$ Chaotic field theory slides

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Mephistopheles knocks at Faust's door and says, “Du mußt es dreimal sagen!"

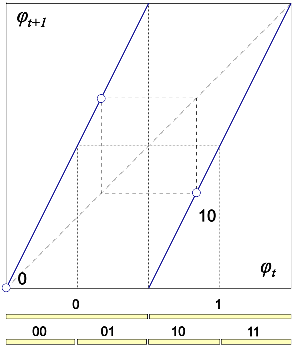
“You have to say it three times"

— Johann Wolfgang von Goethe
Faust I - Studierzimmer 2. Teil

time-evolution formulation

fair coin toss

Bernoulli map



$$\phi_{t+1} = \begin{cases} 2\phi_t \\ 2\phi_t \pmod 1 \end{cases}$$

$\Rightarrow$ fixed point $\overline{0}$, 2-cycle $\overline{01}$, $\cdots$

a coin toss

the essence of deterministic chaos

what is (mod 1) ?

map with integer-valued ‘stretching’ parameter $s \geq 2$:

$$x_{t+1} \; = \; s\, x_t$$

(mod 1) : subtract the integer part $m_t = \lfloor s x_t \rfloor$
so fractional part ϕ_{t+1} stays in the unit interval $[0, 1)$

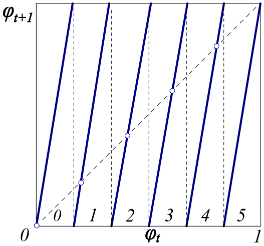
$$\phi_{t+1} = s\phi_t - m_t \,, \qquad \phi_t \in \mathcal{M}_{m_t}$$

m_t takes values in the s-letter alphabet

$$m \in \mathcal{A} = \{0, 1, 2, \cdots , s - 1\}$$

a fair dice throw

slope 6 Bernoulli map



$$\phi_{t+1} = 6\phi_t - m_t\,,\ \phi_t \in \mathcal{M}_{m_t}$$

6-letter alphabet

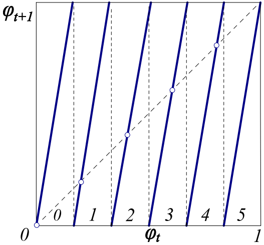
$m_t \in \mathcal{A} = \{0, 1, 2, \cdots, 5\}$

6 subintervals $\{\mathcal{M}_0, \mathcal{M}_1, \cdots, \mathcal{M}_5\}$

what is chaos ?

a fair dice throw

6 subintervals $\{\mathcal{M}_{m_t}\}$, 6^2 subintervals $\{\mathcal{M}_{m_1 m_2}\}, \cdots$



each subinterval contains a periodic point, labeled by $\mathsf{M} = m_1 m_2 \cdots m_n$

$N_n = 6^n - 1$ unstable orbits

definition : chaos is

positive Lyapunov ($\ln s$) - positive entropy ($\frac{1}{n} \ln N_n$)

definition : chaos is

positive Lyapunov ($\ln s$) - positive entropy ($\frac{1}{n} \ln N_n$)

- Lyapunov : how fast is local escape?
- entropy : how many ways of getting back?

$\Rightarrow$ ergodicity

the precise sense in which dice throw
is an example of deterministic chaos

lattice formulation

lattice Bernoulli

recast the time-evolution Bernoulli map

$$\phi_{t+1} = s\phi_t - m_t$$

as 1-step difference equation on the temporal lattice

$$\phi_{t+1} - s\phi_t = -m_t\,, \qquad \phi_t \in [0,1)$$

field ϕ_t, source m_t
on each site t of a 1-dimensional lattice $t \in \mathbb{Z}$

write an n-sites lattice segment as
the field configuration and the symbol block

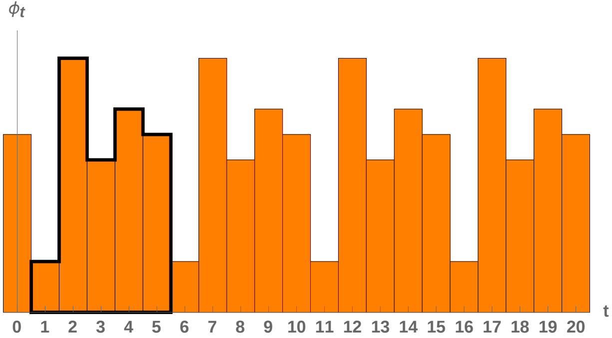
$$\Phi = (\phi_{t+1}, \cdots, \phi_{t+n})\,, \quad \mathsf{M} = (m_{t+1}, \cdots, m_{t+n})$$

'M' for 'marching orders' : come here, then go there, $\cdots$

scalar field theory on 1-dimensional lattice

write a periodic field over n-sites Bravais cell as the field configuration and the symbol block (sources)

$$\Phi = (\phi_{t+1}, \cdots, \phi_{t+n}), \quad \mathsf{M} = (m_{t+1}, \cdots, m_{t+n})$$



'M' for 'marching orders' : come here, then go there, $\cdots$

think globally, act locally

Bernoulli condition at every lattice site t, local in time

$$-\phi_{t+1} + s\phi_t = m_t$$

is enforced by the global equation

$$(-r + s\,1)\,\Phi = \mathsf{M}\,,$$

$[n \times n]$ shift matrix

$$r_{jk} = \delta_{j+1,k}\,, \qquad r = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & & \ddots & \\ & & & 0 & 1 \\ 1 & & & & 0 \end{pmatrix}$$

compares the neighbors

think globally, act locally

solving the lattice Bernoulli system

$$\mathcal{J}\Phi = \mathsf{M}\,,$$

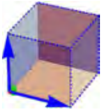
$[n\times n]$ Hill matrix $\qquad \mathcal{J} = -r + s\,1\,,$

is a search for zeros of the function

$$F[\Phi] = \mathcal{J}\Phi - \mathsf{M} = 0$$

the entire global lattice state Φ_{M} is now

a single fixed point $(\phi_1, \phi_2, \cdots, \phi_n)$



in the n-dimensional unit hyper-cube $\qquad \Phi \in [0,1)^n$

orbit stability

Hill matrix

solving a nonlinear

$$F[\Phi] = 0 \quad \text{fixed point condition}$$

with Newton method requires evaluation of the $[n \times n]$

Hill matrix

$$\mathcal{J}_{ij} = \frac{\delta F[\Phi]_i}{\delta \phi_j}$$

what does this global Hill matrix do?

1. fundamental fact !
2. global stability of lattice state Φ, perturbed everywhere

(1)

fundamental fact

(1) fundamental fact

to satisfy the fixed point condition

$$\mathcal{J}\Phi - \mathrm{M} = 0$$

the Hill matrix $\mathcal{J}$

1. stretches the unit hyper-cube $\Phi \in [0,1)^n$ into the n-dimensional fundamental parallelepiped
2. maps each periodic point Φ_M $\Rightarrow$ integer lattice $\mathbb{Z}^n$ point
3. then translate by integers M $\Rightarrow$ into the origin

hence $N_n =$ total $\sharp$ solutions = $\sharp$ integer lattice points within the fundamental parallelepiped

the fundamental fact[1] : Hill determinant counts solutions

$$N_n = \mathrm{Det}\,\mathcal{J}$$

$\sharp$ integer points in fundamental parallelepiped $=$ its volume

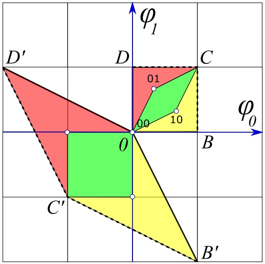
[1] M. Baake et al., J. Phys. A **30**, 3029–3056 (1997).

example : fundamental parallelepiped for $n = 2$

Hill matrix for $s = 2$; unit square basis vectors ; their images :

$$\mathcal{J} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} ; \quad \Phi_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \Phi_{B'} = \mathcal{J}\,\Phi_B = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \cdots ,$$

Bernoulli periodic points of period 2



$N_2 = 3$

fixed point Φ_{00}

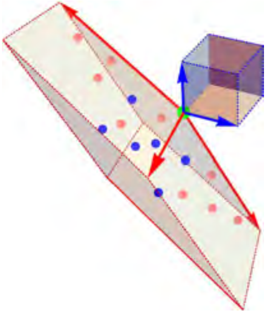
2-cycle Φ_{01}, Φ_{10}

square $[0BCD] \Rightarrow \mathcal{J} \Rightarrow$ fundamental parallelepiped $[0B'C'D']$

fundamental fact for any n

an $n = 3$ example

$\mathcal{J}$ [unit hyper-cube] = [fundamental parallelepiped]



unit hyper-cube $\Phi \in [0, 1)^3$

$n > 3$ cannot visualize

a periodic point $\Rightarrow$ integer lattice point : $\bullet$ on a face, $\bullet$ in the interior

orbit stability

(2) orbit stability vs. temporal stability

Hill matrix

$\mathcal{J}_{ij} = \frac{\delta F[\Phi]_i}{\delta \phi_j}$ stability under global perturbation of the whole orbit

for n large, a huge $[dn \times dn]$ matrix

temporal Jacobian matrix

J propagates initial perturbation n time steps

small $[d \times d]$ matrix

J and $\mathcal{J}$ are related by[2]

Hill's 1886 remarkable formula

$$|\mathrm{Det}\, \mathcal{J}_\mathrm{M}| = |\det\left(\mathbf{1} - J_\mathrm{M}\right)|$$

$\mathcal{J}$ is huge, even ∞-dimensional matrix
J is tiny, few degrees of freedom matrix

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[2] G. W. Hill, Acta Math. **8**, 1–36 (1886).

field theorist's chaos

definition : chaos is

expanding	Hill determinants	$\mathrm{Det}\,\mathcal{J}$
exponential $\sharp$	field configurations	N_n

the precise sense in which
a (discretized) field theory is deterministically chaotic

note : there is no 'time' in this definition

periodic orbit theory

volume of a periodic orbit

Ozorio de Almeida and Hannay[3] 1984 :
$\sharp$ of periodic points is related to a Jacobian matrix by

principle of uniformity

"periodic points of an ergodic system, counted with their natural weighting, are uniformly dense in phase space"

where

'natural weight' of periodic orbit M

$$\frac{1}{|\det(1 - J_{\mathsf{M}})|}$$

[3] A. M. Ozorio de Almeida and J. H. Hannay, J. Phys. A **17**, 3429 (1984).

periodic orbits partition lattice states into neighborhoods

how come Hill determinant $\mathrm{Det}\,\mathcal{J}$ counts periodic points ?

'principle of uniformity' is in[4]

periodic orbit theory

known as the flow conservation sum rule :

$$\sum_{\mathsf{M}} \frac{1}{|\det(1 - J_{\mathsf{M}})|} = \sum_{\mathsf{M}} \frac{1}{|\mathrm{Det}\,\mathcal{J}_{\mathsf{M}}|} = 1$$

sum over periodic points Φ_{M} of period n

state space is divided into

neighborhoods of periodic points of period n

[4] P. Cvitanović, "Why cycle?", in *Chaos: Classical and Quantum*, edited by P. Cvitanović et al. (Niels Bohr Inst., Copenhagen, 2020).

periodic orbit counting

how come a $\mathrm{Det}\,\mathcal{J}$ counts periodic points ?

flow conservation sum rule :

$$\sum_{\Phi_{\mathsf{M}} \in \mathrm{Fix} f^n} \frac{1}{|\mathrm{Det}\,\mathcal{J}_{\mathsf{M}}|} = 1$$

Bernoulli system 'natural weighting' is simple :

the determinant $\mathrm{Det}\,\mathcal{J}_{\mathsf{M}} = \mathrm{Det}\,\mathcal{J}$ the same for all periodic points, whose number thus verifies the fundamental fact

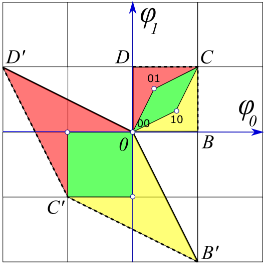
$$N_n = |\mathrm{Det}\,\mathcal{J}|$$

the number of Bernoulli periodic lattice states

$N_n = |\mathrm{Det}\,\mathcal{J}| = s^n - 1$ for any n

remember the fundamental fact?

period 2 example



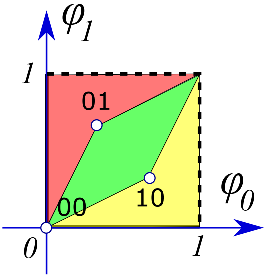
fixed point Φ_{00}
2-cycle Φ_{01}, Φ_{10}

$\mathcal{J}$ [unit hyper-cube] = [fundamental parallelepiped]

look at preimages of the fundamental parallelepiped :

example : lattice states of period 2

unit hypercube, partitioned



fixed point Φ_{00}

2-cycle Φ_{01}, Φ_{10}

flow conservation sum rule

$$\frac{1}{|\mathrm{Det}\,\mathcal{J}_{00}|} + \frac{1}{|\mathrm{Det}\,\mathcal{J}_{01}|} + \frac{1}{|\mathrm{Det}\,\mathcal{J}_{10}|} = 1$$

sum over periodic points Φ_M of period $n = 2$

state space is divided into

neighborhoods of periodic points of period n

Amazing! I did not understand a single word.

—Fritz Haake 1988

zeta function

periodic orbit theory, version (1) : counting lattice states

topological zeta function

$$1/\zeta_{\text{top}}(z) = \exp\left(-\sum_{n=1}^{\infty}\frac{z^n}{n}N_n\right)$$

1. weight $1/n$ as by (cyclic) translation invariance, n lattice states are equivalent
2. zeta function counts orbits, one per each set of equivalent lattice states

Bernoulli topological zeta function

counts orbits, one per each set of lattice states $N_n = s^n - 1$

$$1/\zeta_{\text{top}}(z) = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} N_n\right) = \frac{1 - sz}{1 - z}$$

numerator $(1 - sz)$ says that Bernoulli orbits are built from s fundamental primitive lattice states,

the fixed points $\{\phi_0, \phi_1, \cdots, \phi_{s-1}\}$

every other lattice state is built from their concatenations and repeats.

solved!

this is ‘periodic orbit theory’

And if you don’t know, now you know

think globally, act locally - summary

the problem of enumerating and determining all lattice states stripped to its essentials :

1. each solution is a zero of the global fixed point condition

$$F[\Phi] = 0$$

2. global stability : the Hill matrix

$$\mathcal{J}_{ij} = \frac{\delta F[\Phi]_i}{\delta \phi_j}$$

3. fundamental fact : the number of period-n orbits

$$N_n = |\mathrm{Det}\, \mathcal{J}|$$

4. zeta function $1/\zeta_{\mathrm{top}}(z)$: all predictions of the theory

next : a kicked rotor

Du mußt es dreimal sagen!

— Mephistopheles

1. what this is about
2. coin toss
3. **kicked rotor**
4. spatiotemporal cat
5. bye bye, dynamics