Second Menu of Projects (for Project IV)

Calculating g and α to higher accuracy for the period-doubling route to chaos. In this project, you will determine α and g to higher accuracy than is done in Chapter 5 of the class notes. First, please extend the Taylor-series calculation presented there to higher orders in z. How does the value of α and the form of g depend on the number of terms that are kept in the Taylor series? Does the process appear to converge? Then, we would like you to compare your results for g to those obtained when you use Feigenbaum's result that g(z) can be obtained by repeatedly iterating any function f(z) with a quadratic maximum at z=0 and rescaling appropriately. Specifically, Feigenbaum showed that g(x) can be written as the limit:

$$g(z) = \lim_{n\to\infty} (-\alpha)^n f^{2^n}\left(\frac{z}{(-\alpha)^n}\right) \qquad (5.8?)$$

where the map f is iterated at the parameter value $r = r_\infty$. (We say a bit about this method in the Appendix. For more detailed expositions, see, for example, the book by Hilborn.) In practice, one plugs in the known value of α (α is determined by equation (5.8) because the limit exists only for that particular value), and then iterates some large but finite number of times to obtain an approximate result for g. How does your result for g depend on the number of iterations you perform on the map function? How does it compare to the results you obtained using the Taylor series method?

Calculating δ for the period-doubling route to chaos.
In this project, you will calculate δ, which is a measure of how many iterations are required to determine that the parameter value r is not exactly equal to r_∞. We recommend that you consult the references to see how to do this (Hilborn, section 5.7 has a heuristic discussion; the rigorous derivations are in M. J. Feigenbaum, J. Stat. Phys. **19**, 25 (1978); **21**, 669 (1979)); here we say just a few words to give an overview of the procedure.

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To calculate δ you need to how the map function itself evolves under iteration. In particular, you need to figure out which deviations of the map function f(z) from the universal function g(z) are the ones that cause it to diverge from the universal function as it is iterated. Then, you will consider functions which are close to the fixed function and express the evolution in terms of equations which are linearized about the fixed function g(z). You will find that the deviations can be described in terms of an eigenfunction and an eigenvalue, the latter of which is δ. Since this eigenfunction grows by the factor δ each time the map is iterated, whereas all the other deviations shrink under iteration, it determines the deviations from criticality and hence the convergence of the sequence of r-values.

Periodic and Non-periodic Orbits in a Central Potential. By doing the integral in Eq (6.8) one can see whether for a given potential and given values of E and L Φ is a rational number. If it is rational, then the orbit closes, if not the orbit never repeats itself. One might then ask the question about whether there are any forms of the potential V(r) which permit the orbit to be closed for all E and L. However, once V(r) is fixed we know that Φ is a function of E and L, which we then write as $\Phi(E,L)$. Some applications of theorems derived from calculus indicate that the function is continuous. Hence it can only be rational by being constant, independent of E and L.

It turns out that $\Phi(E,L)$ is constant only for very special potentials, those which vary as a power of the distance:

$$V(r) = -\alpha\, r^{-\alpha}$$

One can recognize two familiar special cases, $\alpha=1$, which is the attractive gravitational force, and $\alpha=-2$, which is the harmonic oscillator. Compute the value of $\Phi(E,L)$ for these two cases and find out whether the orbit closes. Then do the same for $\alpha=3$. Plot up a few orbits to show what is going on.

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Notice that the integrals involved must be computed carefully because of their singularities at the two endpoints. To do them accurately, one should first compute r_{min} and r_{max} and then make a transformation like that in Problem 6.4 to eliminate the singularities at the endpoints.

Period-doubling bifurcation sequence for motion in a double-well potential. Compare Figure 7.1 to Figure 2.2, the possible values of x for different values of r in the logistic map. Determine whether the double-well system has a period-doubling bifurcation sequence, and if so, whether it is similar to that of the logistic map. (You will need to define what you mean by the word "similar" here.) How does your answer depend on the choice of parameters of the double-well system?

Mapping Regions for a Double Well System.

In this project, you will investigate the evolution of phase space regions for the damped driven motion of a particle in the double well. The idea is to take a whole bunch of points (at least a thousand) bunched into a fairly small area, propagate them all forward in time using the equations of motion, and see what the resulting region looks like. Can you think of a way of estimating the area of the resulting region? In any case, try to determine if the area seems to behave as expected from equation (7.8)

According to equation (7.8), how the area of the region changes in time should be independent of the value of F_0. However, the shape of the region does depend on F_0. Investigate the evolution of the shapes of the regions in the different regimes. This project is quite open-ended, and we expect you to do interesting and imaginative work.

Mapping Phase Space Regions for The Standard Map. In this project, you will characterize the evolution of areas in phase space for the standard map (8.10). Take a whole

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bunch of points (like maybe a thousand) bunched into a fairly small area, and iterate them all by (8.10), and see what the resulting region looks like. Typically, it will be a distortion of the original region, but it should have the same area. Try to determine if the area seems to be preserved after several iterations. Then see what you can say about how the shape of the area evolves as the map is iterated. Do this also with mapping (6.3), and compare the two cases.

Dissipative Standard Map. In this project, you will study the map defined by:

$$\theta_{j+1} = \theta_j + p_{j+1},$$
$$p_{j+1} = bp_j + (1-b)\Omega - k\sin(\theta_j),$$

where $0 \le b \le 1$. This map, called the dissipative standard map, is identical to the standard map when the parameter b is equal to 1. By calculating how phase space areas evolve, you should be able to understand the physical significance of the parameter b. What is the physical significance of the parameter Ω? Calculate the phase space portraits for this map, and characterize the different types of behavior.

The Undamped Driven Pendulum. This project concerns the dynamics of the undamped driven pendulum, described by the equation of motion:

$$\frac{d^2\theta}{dt^2} = -\sin\theta + F_0 \sin(\omega t).$$

Make some phase space plots for this system, being careful to solve the equations of motion accurately enough so that the orbits appear to close when $F_0 = 0$. How does increasing F_0 change the behavior? If you plot the (p,θ) pairs at discrete times separated by the drive period, how do these phase space plots compare to those for the standard map?

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Lyapunov Index for the Lorenz Equations. Recall from Chapter 7 the Lorenz equations:

$$\frac{d}{dt}x = p(y - x)$$
$$\frac{d}{dt}y = -xz + rx - y$$
$$\frac{d}{dt}z = xy - bz$$
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Estimate λ for the Lorenz equations for the parameter values p = 10, b = 8/3, and different values of r. What do you think is the error in your estimate?

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