# Required Project I:     The logistic map, Part I

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| Note: If you are asked to write an applet, you are expected to put a working version of it into the drop box. You are also expected to turn in hard copy solutions which include listings of all the applets. |

The logistic map is defined by the rule:

$$x_{j+1} = f(x_j) \ . \tag{RP1.1}$$

with

$$f(x) = r*x*(1-x) \ . \tag{RP1.2}$$

RP1.1 : Producing the Iterations. Write an applet (perhaps by modifying the program "Iterate" in Chapter 2) that, given a value of r and a starting value $x_0$, prints out on an applet a large number of the values $x_1$, $x_2$, .....

Now modify the applet and use the System.out.println() method to print a hundred values out onto the Java Console. Cut and paste the $x_j$ values from the Java Console for one value of r into your project writeup.

We will be restricting x to the range $0 \le x \le 1$, and the parameter r to the range $0 \le r \le 4$ . What happens when x and r are outside these ranges?

RP1.2: Long-Time Solutions .

RP1.2a: After many iterations, what typically happens to x for the following values of r: 1.5, 2.5, and 3.4. These are relatively easy. What about for r equal to 3.55? 3.65? You should try several different initial conditions, to be sure that the ones you chose were not in any way 'special.' Be sure to include numerical values of x in your answers.

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RP1.2b: For r = 3.65, find the range of values of x which can appear after many iterations starting from some typical value of x between zero and one. What range of x-values appears when r = 4?

RP1.2c: Write an applet that draws a "bifurcation diagram" ¾ a graph showing the possible long-time values of x as a function of r (analogous to figure 2.2). Make the program work so that the user may choose which interval in r is plotted. Be sure to put the applet in the drop box.

RP1.3: Finding Fixed Points (analytic work).

RP1.3a: Find the fixed point values for the logistic map $f(x) = rx(1-x)$.

RP1.3b: Follow the fixed point (at $x^* = 1 - r^{-1}$) of the logistic map as r changes from 0 to 4. When is this fixed point stable? What is the range of values of the Floquet multiplier $\Lambda$ in the region of stability? What happens at the borderline case, $f'(x^*) = \pm 1$?

RP1.3c: Two-Cycles. Find analytically the x-values for the elements of the cycle of length two for the map $f(x) = rx(1-x)$ for $0 < r < 4$. Draw a rough picture of your results (i.e. the two x values as a function of r). What is $r_2$*, the lowest positive value of r for which the two-cycle is present? For r < $r_2$*, what is the long-run behavior of the iterates? Just above $r_2$*, what is the long-run behavior? Show both algebraically and graphically that the logistic map has no more than one cycle of length two for r between 0 and 4.

RP1.3d: For what range in r is the period-2 orbit stable?

RP1.4: Finding Fixed Points (numerical work).

RP1.4a: Two-cycles. Now we want you to calculate numerically how the elements of the cycle of length 2 depend on r for r just above $r_2$*, the value of r at which the two-cycle first appears. You will compare the results to the analytic calculation you just did, and then you will modify this program and look at the behavior near onset of 3- and 4-cycles (which are much more complicated to calculate analytically).

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Write an applet which computes the elements of the 2-cycles for r close to $r_2$* (using the Newton-Raphson or secant method). Make a table of these elements, and verify the approximation

$$x_{\pm} = a \pm b\sqrt{r - r_2 *} + c(r - r_2 *) + O(r - r_2 *)^{3/2},$$

for the two-cycle elements $x_{\pm}$, by giving estimates of a, b, and c.

(Hint: To find a, b, and c from your table, you might try to fit $x_+ + x_-$ and $x_+ - x_-$ as functions of r. To do this, plot them against r and try to determine the form of the function. If it is a line, measure its slope. For other functions you will have to be a little more clever.) Show that your numerical result is consistent with the analytic result you obtained in RP1.3c.

RP1.4b. Four-cycles. When the two-cycle becomes unstable, a four-cycle appears. Write an applet that computes the elements of the 4-cycles for values of r just above value $r_4$* where the 4-cycle first appears. You will find that the orbit elements come in pairs, two "emerging" from $x_+$ of the two-cycle and two from $x_-$. If you label the values on the 4-cycle $x_{++}$, $x_{+-}$, $x_{-+}$, and $x_{--}$, then you should find that

$$x_{\pm +} = a_{\pm} + b_{\pm}(r - r_4 *)^{\gamma_+} + O(r - r_4 *)^1, \quad x_{\pm -} = a_{\pm} - b_{\pm}(r - r_4 *)^{\gamma_-} + O(r - r_4 *)^1,$$

where both $\gamma_+$ and $\gamma_-$ are less than one, and $a_{\pm}$ and $b_{\pm}$ are constants.

Find the exponents $\gamma_+$ and $\gamma_-$. Compare your results for the four-cycle to the behavior of the two-cycles near their onset.

RP1.4c. Three-cycles. Write an applet that finds the elements of the cycles of length three in the logistic map for r=3.95 using the Newton-Raphson (or secant) method. How many period-3 cycles are there? What are the elements of each cycle? Are these cycles stable or unstable? Answer the same questions for r=4. As r decreases what happens to these three-cycles? At which value of r?

RP1.4d: Period-doubling bifurcations. Find analytically and make a graph of the Floquet multiplier of the 2-cycle of the logistic map between $r_2$*, where the two cycle first appears, and $r_4$*, where the 2-cycle bifurcates to a 4-cycle. (The doubling of the length of the cycle is called a bifurcation.) Write an applet that plots the Floquet multiplier as a function of r for the $2^n$ times iterated map

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between $r_{2^n}$*, where the $2^n$ cycle is born, and $r_{2^{n+1}}$*, where this cycle becomes unstable, for n=2 and 3. Can you formulate a general rule for the stabilities of the $2^n$ and $2^{n+1}$ cycles at the bifurcation point where the $2^n$ cycle becomes unstable and the $2^{n+1}$ cycle is born?

RP1.5: Chaos. Let us imagine that we compare two different developments of population at r = 4 starting from almost the same values of x. One population starts with the value $x_0$, the other with the value $y_0 = x_0 + \varepsilon$, where ε is very small. Then the populations develop through many iterations. At the beginning the populations remain close to one another. Then, after some large number of steps they begin to be more and more different. In fact after J steps the values $x_J$ and $y_J$ which respectively developed from $x_0$ and $y_0$ have begun to differ by an amount of order unity. Question: How does $x_J - y_J$ depend upon ε and J? (You may answer this question by writing a program, or you can take advantage of the variable change discussed at the end of Chapter 3 and find a solution analytically.)

You have noticed that $x_J$ has a sensitivity to the value of $x_0$ which grows very rapidly as a function of J. This kind of sensitivity to initial conditions is a hallmark of chaotic systems.

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