Chapter 18

Trace formulas

The trace formula is not a formula, it is an idea.
—Martin Gutzwiller

Dynamics is posed in terms of local equations, but the ergodic averages require global information. How can we use a local description of a flow to learn something about the global behavior? In chapter 17 we have related global averages to the eigenvalues of appropriate evolution operators. Here we show that the traces of evolution operators can be evaluated as integrals over Dirac delta functions, and in this way the spectra of evolution operators become related to periodic orbits. If there is one idea that one should learn about chaotic dynamics, it happens in this chapter, and it is this: there is a fundamental local $\leftrightarrow$ global duality which says that

the spectrum of eigenvalues is dual to the spectrum of periodic orbits

For dynamics on the circle, this is called Fourier analysis; for dynamics on well-tiled manifolds, Selberg traces and zetas; and for generic nonlinear dynamical systems the duality is embodied in the trace formulas that we will now derive. These objects are to dynamics what partition functions are to statistical mechanics.

The above phrasing is a bit too highfalutin, so it perhaps pays to go again through the quick sketch of sects. 1.5 and 1.6. We have a state space that we would like to tessellate by periodic orbits, one short orbit per neighborhood, as in figure 18.1 (a). How big is the neighborhood of a given cycle?

Along stable directions neighbors of the periodic orbit get closer with time, so we only have to keep track of those who are moving away along the unstable directions. The fraction of those who remain in the neighborhood for one recurrence time T_p is given by the overlap ratio along the initial sphere and the returning ellipsoid, figure 18.1 (b), and along the expanding eigen-direction $\mathbf{e}^{(i)}$ of $J_p(x)$ this is given by the expanding Floquet multiplier $1/|\Lambda_{p,i}|$. A bit more thinking

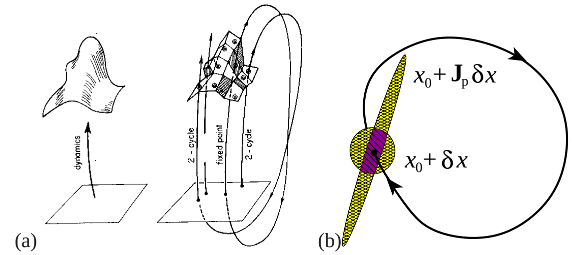

Figure 18.1: (a) Smooth dynamics tesselated by the skeleton of periodic points, together with their linearized neighborhoods. (b) Jacobian matrix J_p maps spherical neighborhood of $x_0 \to$ ellipsoidal neighborhood time T_p later, with the overlap ratio along the expanding eigdirection $\mathbf{e}^{(i)}$ of $J_p(x)$ given by the expanding eigenvalue $1/|\Lambda_{p,i}|$.

leads to the conclusion that one also cares about how long it takes to return (the long returns contributing less to the time averages), so the weight t_p of the p-neighborhood $t_p = e^{-sT_p}/|\Lambda_p|$ decreases exponentially both with the shortest recurrence period and the product (5.7) of expanding Floquet multipliers $\Lambda_p = \prod_e \Lambda_{p,e}$. With emphasis on *expanding* - the flow could be a 60,000-dimensional dissipative flow, and still the neighborhood is defined by the handful of expanding eigen-directions. Now the long-time average of a physical observable -let us say power D dissipated by viscous friction of a fluid flowing through a pipe- can be estimated by its mean value (17.6) D_p/T_p computed on each neighborhood, and weighted by the above estimate

$$\langle D \rangle \approx \sum_p \frac{D_p}{T_p} \frac{e^{-sT_p}}{|\Lambda_p|} \,.$$

Wrong in detail, this estimate is the crux of many a *Phys. Rev. Letter*, and in its essence the key result of this chapter, the 'trace formula.' Here we redo the argument in a bit greater depth, and derive the correct formula (20.20) for a long time average $\langle D \rangle$ as a weighted sum over periodic orbits. It will take three chapters, but it is worth it - the reward is an *exact* (i.e., not heuristic) and highly convergent and controllable formula for computing averages over chaotic flows.

18.1 A trace formula for maps

Our extraction of the spectrum of $\mathcal{L}$ commences with the evaluation of the trace. As the case of discrete time mappings is somewhat simpler, we first derive the trace formula for maps, and then, in sect. 18.2, for flows. The final formula (18.23) covers both cases.

To compute an expectation value using (17.21) we have to integrate over all the values of the kernel $\mathcal{L}^n(x,y)$. Were $\mathcal{L}^n$ a matrix sum over its matrix elements would be dominated by the leading eigenvalue as $n \to \infty$ (we went through the argument in some detail in sect. 15.1). As the trace of $\mathcal{L}^n$ is also dominated by the leading eigenvalue as $n \to \infty$, we might just as well look at the trace for which we have a very explicit formula exercise 15.3

$$\mathrm{tr}\,\mathcal{L}^n = \int dx\,\mathcal{L}^n(x,x) = \int dx\,\delta(x - f^n(x))\,e^{\beta\cdot A^n(x)}\,. \tag{18.1}$$

On the other hand, by its matrix motivated definition, the trace is the sum over eigenvalues,

$$\mathrm{tr}\,\mathcal{L}^n = \sum_{\alpha=0}^{\infty} e^{s_\alpha n}\,. \tag{18.2}$$

We find it convenient to write the eigenvalues as exponents e^{s_α} rather than as multipliers λ_α, and we assume that spectrum of $\mathcal{L}$ is discrete, $s_0, s_1, s_2, \cdots$, ordered so that $\mathrm{Re}\ s_\alpha \geq \mathrm{Re}\,s_{\alpha+1}$.

For the time being we choose not to worry about convergence of such sums, ignore the question of what function space the eigenfunctions belong to, and compute the eigenvalue spectrum without constructing any explicit eigenfunctions. We shall revisit these issues in more depth in chapter 23, and discuss how lack of hyperbolicity leads to continuous spectra in chapter 24.

18.1.1 Hyperbolicity assumption

We have learned in sect. 16.2 how to evaluate the delta-function integral (18.1). section 16.2

According to (16.8) the trace (18.1) picks up a contribution whenever $x - f^n(x) = 0$, i.e., whenever x belongs to a periodic orbit. For reasons which we will explain in sect. 18.2, it is wisest to start by focusing on discrete time systems. The contribution of an isolated prime cycle p of period n_p for a map f can be evaluated by restricting the integration to an infinitesimal open neighborhood $\mathcal{M}_p$ around the cycle,

$$\begin{aligned}\mathrm{tr}\,{}_p\mathcal{L}^{n_p} &= \int_{\mathcal{M}_p} dx\,\delta(x - f^{n_p}(x)) \\ &= \frac{n_p}{\left|\det\left(\mathbf{1} - M_p\right)\right|} = n_p \prod_{i=1}^{d} \frac{1}{|1-\Lambda_{p,i}|}\,.\end{aligned} \tag{18.3}$$

For the time being we set here and in (16.9) the observable $e^{\beta A_p} = 1$. Periodic orbit Jacobian matrix M_p is also known as the *monodromy matrix*, and its eigenvalues $\Lambda_{p,1}, \Lambda_{p,2}, \ldots, \Lambda_{p,d}$ as the Floquet multipliers. section 5.1.2

We sort the eigenvalues $\Lambda_{p,1}, \Lambda_{p,2}, \ldots, \Lambda_{p,d}$ of the p-cycle $[d\times d]$ monodromy matrix M_p into expanding, marginal and contracting sets $\{e, m, c\}$, as in (5.6). As the integral (18.3) can be evaluated only if M_p has no eigenvalue of unit magnitude, we assume that no eigenvalue is marginal (we shall show in sect. 18.2 that the longitudinal $\Lambda_{p,d+1} = 1$ eigenvalue for flows can be eliminated by restricting the consideration to the transverse monodromy matrix M_p), and factorize the trace (18.3) into a product over the expanding and the contracting eigenvalues

$$\left|\det\left(\mathbf{1} - M_p\right)\right|^{-1} = \frac{1}{|\Lambda_p|}\prod_e \frac{1}{1-1/\Lambda_{p,e}} \prod_c \frac{1}{1-\Lambda_{p,c}}\,, \tag{18.4}$$

where $\Lambda_p = \prod_e \Lambda_{p,e}$ is the product of expanding eigenvalues. Both $\Lambda_{p,c}$ and $1/\Lambda_{p,e}$ are smaller than 1 in absolute value, and as they are either real or come in complex conjugate pairs we are allowed to drop the absolute value brackets $|\cdots|$ in the above products.

The *hyperbolicity assumption* requires that the stabilities of all cycles included in the trace sums be exponentially bounded away from unity:

$$\begin{aligned}|\Lambda_{p,e}| &> e^{\lambda_e T_p} && \text{any } p, \text{any expanding } |\Lambda_{p,e}| > 1 \\ |\Lambda_{p,c}| &< e^{-\lambda_c T_p} && \text{any } p, \text{any contracting } |\Lambda_{p,c}| < 1\,,\end{aligned} \tag{18.5}$$

where $\lambda_e, \lambda_c > 0$ are strictly positive bounds on the expanding, contracting cycle Lyapunov exponents. If a dynamical system satisfies the hyperbolicity assumption (for example, the well separated 3-disk system clearly does), the $\mathcal{L}^t$ spectrum will be relatively easy to control. If the expansion/contraction is slower than exponential, let us say $|\Lambda_{p,i}| \sim {T_p}^2$, the system may exhibit "phase transitions," and the analysis is much harder - we shall discuss this in chapter 24.

Example 18.1 ***Elliptic stability.*** *Elliptic stability, i.e., a pair of purely imaginary exponents $\Lambda_m = e^{\pm i\theta}$ is excluded by the hyperbolicity assumption. While the contribution of a single repeat of a cycle*

$$\frac{1}{(1-e^{i\theta})(1-e^{-i\theta})} = \frac{1}{2(1-\cos\theta)} \tag{18.6}$$

does not make (16.9) diverge, if $\Lambda_m = e^{i2\pi p/r}$ is rth root of unity, $1/\left|\det\left(\mathbf{1} - M_p^r\right)\right|$ diverges. For a generic θ repeats $\cos(r\theta)$ behave badly and by ergodicity $1-\cos(r\theta)$ is arbitrarily small, $1-\cos(r\theta) < \epsilon$, infinitely often. This goes by the name of "small divisor problem," and requires a separate treatment.

It follows from (18.4) that for long times, $t = rT_p \to \infty$, only the product of expanding eigenvalues matters, $\left|\det\left(\mathbf{1} - M_p^r\right)\right| \to |\Lambda_p|^r$. We shall use this fact to motivate the construction of dynamical zeta functions in sect. 19.3. However, for evaluation of the full spectrum the exact cycle weight (18.3) has to be kept.

18.1.2 A classical trace formula for maps

If the evolution is given by a discrete time mapping, and all periodic points have Floquet multipliers $|\Lambda_{p,i}| \neq 1$ strictly bounded away from unity, the trace $\mathcal{L}^n$ is given by the sum over all *periodic points i* of period n:

$$\mathrm{tr}\,\mathcal{L}^n = \int dx\,\mathcal{L}^n(x,x) = \sum_{x_i\in \mathrm{Fix} f^n} \frac{e^{\beta\cdot A_i}}{|\det\left(\mathbf{1}-M^n(x_i)\right)|}\,. \tag{18.7}$$

Here $\mathrm{Fix}\,f^n = \{x : f^n(x) = x\}$ is the set of all periodic points of period n, and A_i is the observable (17.5) evaluated over n discrete time steps along the cycle to which the periodic point x_i belongs. The weight follows from the properties of the Dirac delta function (16.8) by taking the determinant of $\partial_i(x_j - f^n(x)_j)$. If a trajectory retraces itself r times, its monodromy matrix is M_p^r, where M_p is the $[d\times d]$ monodromy matrix (4.6) evaluated along a single traversal of the prime cycle p. As we saw in (17.5), the integrated observable A^n is additive along the cycle: If a prime cycle p trajectory retraces itself r times, $n = rn_p$, we obtain A_p repeated r times, $A_i = A^n(x_i) = rA_p$, $x_i \in \mathcal{M}_p$.

A prime cycle is a single traversal of the orbit, and its label is a non-repeating symbol string. There is only one prime cycle for each cyclic permutation class. For example, the four periodic points $\overline{0011} = \overline{1001} = \overline{1100} = \overline{0110}$ belong to the same prime cycle $p = 0011$ of length 4. As both the stability of a cycle and the weight A_p are the same everywhere along the orbit, each prime cycle of length n_p contributes n_p terms to the sum, one for each periodic point. Hence (18.7) can be rewritten as a sum over all prime cycles and their repeats chapter 11

$$\mathrm{tr}\,\mathcal{L}^n = \sum_p n_p \sum_{r=1}^{\infty} \frac{e^{r\beta\cdot A_p}}{\left|\det\left(\mathbf{1}-M_p^r\right)\right|}\delta_{n,n_p r}\,, \tag{18.8}$$

with the Kronecker delta $\delta_{n,n_p r}$ projecting out the periodic contributions of total period n. This constraint is awkward, and will be more awkward still for the continuous time flows, where it would yield a series of Dirac delta spikes. In both cases a Laplace transform rids us of the time periodicity constraint.

In the sum over all cycle periods,

$$\sum_{n=1}^{\infty} z^n \mathrm{tr}\,\mathcal{L}^n = \mathrm{tr}\,\frac{z\mathcal{L}}{1-z\mathcal{L}} = \sum_p n_p \sum_{r=1}^{\infty} \frac{z^{n_p r} e^{r\beta\cdot A_p}}{\left|\det\left(\mathbf{1}-M_p^r\right)\right|}\,, \tag{18.9}$$

the constraint $\delta_{n,n_p r}$ is replaced by weight z^n. Such discrete time Laplace transform of $\mathrm{tr}\,\mathcal{L}^n$ is usually referred to as a "generating function." Why this transform? We are actually not interested in evaluating the sum (18.8) for any particular fixed period n; what we are interested in is the long time $n \to \infty$ behavior. The transform trades in the large time n behavior for the small z behavior. Expressing the trace as in (18.2), in terms of the sum of the eigenvalues of $\mathcal{L}$, we obtain the *trace formula for maps*:

$$\sum_{\alpha=0}^{\infty} \frac{ze^{s_\alpha}}{1-ze^{s_\alpha}} = \sum_p n_p \sum_{r=1}^{\infty} \frac{z^{n_p r}\, e^{r\beta\cdot A_p}}{\left|\det\left(\mathbf{1}-M_p^r\right)\right|}\,. \tag{18.10}$$

This is our second example of the duality between the spectrum of eigenvalues and the spectrum of periodic orbits, announced in the introduction to this chapter. (The first example was the topological trace formula (15.10).)

fast track:
sect. 18.2, p. 355

Example 18.2 ***A trace formula for transfer operators:*** *For a piecewise-linear map (17.17), we can explicitly evaluate the trace formula. By the piecewise linearity and the chain rule $\Lambda_p = \Lambda_0^{n_0}\Lambda_1^{n_1}$, where the cycle p contains n_0 symbols 0 and n_1 symbols 1, the trace (18.7) reduces to*

$$\mathrm{tr}\,\mathcal{L}^n = \sum_{m=0}^{n} \binom{n}{m} \frac{1}{|1-\Lambda_0^m\Lambda_1^{n-m}|} = \sum_{k=0}^{\infty}\left(\frac{1}{|\Lambda_0|\Lambda_0^k} + \frac{1}{|\Lambda_1|\Lambda_1^k}\right)^n\,, \tag{18.11}$$

with eigenvalues

$$e^{s_k} = \frac{1}{|\Lambda_0|\Lambda_0^k} + \frac{1}{|\Lambda_1|\Lambda_1^k}\,. \tag{18.12}$$

As the simplest example of spectrum for such dynamical system, consider the symmetric piecewise-linear 2-branch repeller (17.17) for which $\Lambda = \Lambda_1 = -\Lambda_0$. In this case all odd eigenvalues vanish, and the even eigenvalues are given by $e^{s_k} = 2/\Lambda^{k+1}$, k even. exercise 16.7

Asymptotically the spectrum (18.12) is dominated by the lesser of the two fixed point slopes $\Lambda = \Lambda_0$ (if $|\Lambda_0| < |\Lambda_1|$, otherwise $\Lambda = \Lambda_1$), and the eigenvalues e^{s_k} fall off exponentially as $1/\Lambda^k$, dominated by the single less unstable fixed-point. example 23.1

For $k = 0$ this is in agreement with the explicit transfer matrix (17.19) eigenvalues (17.20). The alert reader should experience anxiety at this point. Is it not true that we have already written down explicitly the transfer operator in (17.19), and that it is clear by inspection that it has only one *eigenvalue $e^{s_0} = 1/|\Lambda_0| + 1/|\Lambda_1|$? The example at hand is one of the simplest illustrations of necessity of defining the space that the operator acts on in order to define the spectrum. The transfer operator (17.19) is the correct operator on the space of functions piecewise constant on the state space partition $\{\mathcal{M}_0, \mathcal{M}_1\}$; on this space the operator indeed has only the eigenvalue e^{s_0}. As we shall see in example 23.1, the full spectrum (18.12) corresponds to the action of the transfer operator on the space of real analytic functions.*

The Perron-Frobenius operator trace formula for the piecewise-linear map (17.17) follows from (18.9)

$$\mathrm{tr}\,\frac{z\mathcal{L}}{1-z\mathcal{L}} = \frac{z\left(\frac{1}{|\Lambda_0-1|} + \frac{1}{|\Lambda_1-1|}\right)}{1-z\left(\frac{1}{|\Lambda_0-1|} + \frac{1}{|\Lambda_1-1|}\right)}\,, \tag{18.13}$$

verifying the trace formula (18.10).

18.2 A trace formula for flows

Amazing! I did not understand a single word.
—Fritz Haake

(R. Artuso and P. Cvitanović)

Our extraction of the spectrum of $\mathcal{L}^t$ commences with the evaluation of the trace

$$\mathrm{tr}\,\mathcal{L}^t = \mathrm{tr}\,e^{\mathcal{A}t} = \int dx\,\mathcal{L}^t(x,x) = \int dx\,\delta\Big(x - f^t(x)\Big)\,e^{\beta\cdot A^t(x)}\,. \tag{18.14}$$

We are not interested in any particular time t, but into the long-time behavior as $t \to \infty$, so we need to transform the trace from the "time domain" into the "frequency domain." A generic flow is a semi-flow defined forward in time, so the appropriate transform is a Laplace rather than Fourier.

For a continuous time flow, the Laplace transform of an evolution operator yields the resolvent (16.31). This is a delicate step, since the evolution operator becomes the identity in the $t \to 0^+$ limit. In order to make sense of the trace we regularize the Laplace transform by a lower cutoff ϵ smaller than the period of any periodic orbit, and write

$$\int_\epsilon^\infty dt\,e^{-st}\,\mathrm{tr}\,\mathcal{L}^t = \mathrm{tr}\,\frac{e^{-(s-\mathcal{A})\epsilon}}{s-\mathcal{A}} = \sum_{\alpha=0}^{\infty}\frac{e^{-(s-s_\alpha)\epsilon}}{s-s_\alpha}\,, \tag{18.15}$$

where $\mathcal{A}$ is the generator of the semigroup of dynamical evolution, see sect. 16.5. Our task is to evaluate $\mathrm{tr}\,\mathcal{L}^t$ from its explicit state space representation.

18.2.1 Integration along the flow

As any pair of nearby points on a cycle returns to itself exactly at each cycle period, the eigenvalue of the Jacobian matrix corresponding to the eigenvector along the flow necessarily equals unity for all periodic orbits. Hence for flows the trace integral $\mathrm{tr}\,\mathcal{L}^t$ requires a separate treatment for the longitudinal direction. To evaluate the contribution of an isolated prime cycle p of period T_p, restrict the integration to an infinitesimally thin tube $\mathcal{M}_p$ enveloping the cycle (see figure 1.12), and consider a local coordinate system with a longitudinal coordinate $dx_\parallel$ along the direction of the flow, and $d-1$ transverse coordinates $x_\perp$,

section 5.2.1

$$\mathrm{tr}_p\,\mathcal{L}^t = \int_{\mathcal{M}_p} dx_\perp dx_\parallel\,\delta\Big(x_\perp - f_\perp^t(x)\Big)\,\delta\Big(x_\parallel - f^t(x_\parallel)\Big)\ . \tag{18.16}$$

(we set $\beta = 0$ in the $\exp(\beta\cdot A^t)$ weight for the time being). Pick a point on the prime cycle p, and let

$$v(x_\parallel) = \left(\sum_{i=1}^{d} v_i(x)^2\right)^{1/2} \tag{18.17}$$

be the magnitude of the tangential velocity at any point $x = (x_\parallel, 0, \cdots, 0)$ on the cycle p. The velocity $v(x)$ must be strictly positive, as otherwise the orbit would stagnate for infinite time at $v(x) = 0$ points, and that would get us nowhere.

As $0 \le \tau < T_p$, the trajectory $x_\parallel(\tau) = f^\tau(x_p)$ sweeps out the entire cycle, and for larger times $x_\parallel$ is a cyclic variable of periodicity T_p,

$$x_\parallel(\tau) = x_\parallel(\tau + rT_p) \qquad r = 1, 2, \cdots \tag{18.18}$$

We parametrize both the longitudinal coordinate $x_\parallel(\tau)$ and the velocity $v(\tau) = v(x_\parallel(\tau))$ by the flight time τ, and rewrite the integral along the periodic orbit as

$$\oint_p dx_\parallel\,\delta\Big(x_\parallel - f^t(x_\parallel)\Big) = \oint_p d\tau\,v(\tau)\,\delta(x_\parallel(\tau) - x_\parallel(\tau + t))\,. \tag{18.19}$$

By the periodicity condition (18.18) the Dirac δ function picks up contributions for $t = rT_p$, so the Laplace transform can be split as

$$\int_0^\infty dt\,e^{-st}\,\delta(x_\parallel(\tau) - x_\parallel(\tau + t)) = \sum_{r=1}^{\infty} e^{-sT_p r}\,I_r$$
$$I_r = \int_{-\epsilon}^{\epsilon} dt\,e^{-st}\,\delta\Big(x_\parallel(\tau) - x_\parallel(\tau + rT_p + t)\Big)\,.$$

Taylor expanding and applying the periodicity condition (18.18), we have $x_\parallel(\tau + rT_p + t) = x_\parallel(\tau) + v(\tau)t + \ldots$,

$$I_r = \int_{-\epsilon}^{\epsilon} dt\,e^{-st}\,\delta\Big(x_\parallel(\tau) - x_\parallel(\tau + rT_p + t)\Big) = \frac{1}{v(\tau)}\,,$$

so the remaining integral (18.19) over τ is simply the cycle period $\oint_p d\tau = T_p$. The contribution of the longitudinal integral to the Laplace transform is thus

$$\int_0^\infty dt\,e^{-st}\oint_p dx_\parallel\,\delta\Big(x_\parallel - f^t(x_\parallel)\Big) = T_p\sum_{r=1}^{\infty} e^{-sT_p r}\,. \tag{18.20}$$

This integration is a prototype of what needs to be done for each marginal direction, whenever existence of a conserved quantity (energy in Hamiltonian flows, angular momentum, translational invariance, etc.) implies existence of a smooth manifold of equivalent (equivariant) solutions of dynamical equations.

18.2.2 Stability in the transverse directions

Think of the $\tau = 0$ point in above integrals along the cycle p as a choice of a particular Poincaré section. As we have shown in sect. 5.3, the transverse Floquet multipliers do not depend on the choice of a Poincaré section, so ignoring the dependence on $x_{\|}(\tau)$ in evaluating the transverse integral in (18.16) is justified. For the transverse integration variables the Jacobian matrix is defined in a reduced Poincaré surface of section $\mathcal{P}$ of fixed $x_{\|}$. Linearization of the periodic flow transverse to the orbit yields

$$\int_{\mathcal{P}} dx_\perp \delta\left(x_\perp - f_\perp^{rT_p}(x)\right) = \frac{1}{\left|\det\left(\mathbf{1} - M_p^r\right)\right|}\,, \tag{18.21}$$

where M_p is the p-cycle $[d-1 \times d-1]$ *transverse* monodromy matrix. As in (18.5) we have to assume hyperbolicity, i.e., that the magnitudes of all transverse eigenvalues are bounded away from unity.

Substitution (18.20), (18.21) in (18.16) leads to an expression for $\mathrm{tr}\,\mathcal{L}^t$ as a sum over all prime cycles p and their repetitions

$$\int_\epsilon^\infty dt\, e^{-st}\,\mathrm{tr}\,\mathcal{L}^t = \sum_p T_p \sum_{r=1}^\infty \frac{e^{r(\beta\cdot A_p - sT_p)}}{\left|\det\left(\mathbf{1} - M_p^r\right)\right|}\,. \tag{18.22}$$

The $\epsilon \to 0$ limit of the two expressions for the resolvent, (18.15) and (18.22), now yields the *classical trace formula for flows*

$$\sum_{\alpha=0}^\infty \frac{1}{s - s_\alpha} = \sum_p T_p \sum_{r=1}^\infty \frac{e^{r(\beta\cdot A_p - sT_p)}}{\left|\det\left(\mathbf{1} - M_p^r\right)\right|}\,. \tag{18.23}$$

exercise 18.1

(If you are worried about the convergence of the resolvent sum, keep the ε regularization.)

This formula is still another example of the duality between the (local) cycles and (global) eigenvalues. If T_p takes only integer values, we can replace $e^{-s} \to z$ throughout, so the trace formula for maps (18.10) is a special case of the trace formula for flows. The relation between the continuous and discrete time cases can be summarized as follows:

$$\begin{aligned} T_p &\leftrightarrow n_p \\ e^{-s} &\leftrightarrow z \\ e^{t\mathcal{A}} &\leftrightarrow \mathcal{L}^n\,. \end{aligned} \tag{18.24}$$

We could now proceed to estimate the location of the leading singularity of $\mathrm{tr}\,(s-\mathcal{A})^{-1}$ by extrapolating finite cycle length truncations of (18.23) by methods

such as Padé approximants. However, it pays to first perform a simple resummation which converts this divergence of a trace into a *zero* of a spectral determinant. We shall do this in sect. 19.2, but first a brief refresher of how all this relates to the formula for escape rate (1.8) offered in the introduction might help digest the material.

fast track:
sect. 19, p. 362

18.3 An asymptotic trace formula

In order to illuminate the manipulations of sect. 18.1 and relate them to something we already possess intuition about, we now rederive the heuristic sum of sect. 1.5.1 from the exact trace formula (18.10). The Laplace transforms (18.10) or (18.23) are designed to capture the time $\to \infty$ asymptotic behavior of the trace sums. By the hyperbolicity assumption (18.5), for $t = T_p r$ large the cycle weight approaches

$$\left|\det\left(\mathbf{1} - M_p^r\right)\right| \to |\Lambda_p|^r\,, \tag{18.25}$$

where Λ_p is the product of the expanding eigenvalues of M_p. Denote the corresponding approximation to the nth trace (18.7) by

$$\Gamma_n = \sum_i^{(n)} \frac{1}{|\Lambda_i|}\,, \tag{18.26}$$

and denote the approximate trace formula obtained by replacing the cycle weights $\left|\det\left(\mathbf{1} - M_p^r\right)\right|$ by $|\Lambda_p|^r$ in (18.10) by $\Gamma(z)$. Equivalently, think of this as a replacement of the evolution operator (17.23) by a transfer operator (as in example 18.2). For concreteness consider a dynamical system whose symbolic dynamics is complete binary, for example the 3-disk system figure 1.6. In this case distinct periodic points that contribute to the nth periodic points sum (18.8) are labeled by all admissible itineraries composed of sequences of letters $s_i \in \{0, 1\}$:

$$\begin{aligned} \Gamma(z) &= \sum_{n=1}^\infty z^n \Gamma_n = \sum_{n=1}^\infty z^n \sum_{x_i \in \mathrm{Fix} f^n} \frac{e^{\beta\cdot A^n(x_i)}}{|\Lambda_i|} \\ &= z\left\{\frac{e^{\beta\cdot A_0}}{|\Lambda_0|} + \frac{e^{\beta\cdot A_1}}{|\Lambda_1|}\right\} + z^2\left\{\frac{e^{2\beta\cdot A_0}}{|\Lambda_0|^2} + \frac{e^{\beta\cdot A_{01}}}{|\Lambda_{01}|} + \frac{e^{\beta\cdot A_{10}}}{|\Lambda_{10}|} + \frac{e^{2\beta\cdot A_1}}{|\Lambda_1|^2}\right\} \\ &\quad + z^3\left\{\frac{e^{3\beta\cdot A_0}}{|\Lambda_0|^3} + \frac{e^{\beta\cdot A_{001}}}{|\Lambda_{001}|} + \frac{e^{\beta\cdot A_{010}}}{|\Lambda_{010}|} + \frac{e^{\beta\cdot A_{100}}}{|\Lambda_{100}|} + \ldots\right\} \end{aligned} \tag{18.27}$$

Both the cycle averages A_i and the stabilities Λ_i are the same for all points $x_i \in \mathcal{M}_p$ in a cycle p. Summing over repeats of all prime cycles we obtain

$$\Gamma(z) = \sum_p \frac{n_p t_p}{1 - t_p}, \qquad t_p = z^{n_p} e^{\beta \cdot A_p} / |\Lambda_p| \,. \tag{18.28}$$

This is precisely our initial heuristic estimate (1.9). Note that we could not perform such sum over r in the exact trace formula (18.10) as $\left|\det\left(\mathbf{1} - M_p^r\right)\right| \neq \left|\det\left(\mathbf{1} - M_p\right)\right|^r$; the correct way to resum the exact trace formulas is to first expand the factors $1/|1 - \Lambda_{p,i}|$, as we shall do in (19.9). section 19.2

If the weights $e^{\beta A^n(x)}$ are multiplicative along the flow, and the flow is hyperbolic, for given β the magnitude of each $|e^{\beta A^n(x_i)}/\Lambda_i|$ term is bounded by some constant M^n. The total number of cycles grows as 2^n (or as e^{hn}, $h =$ topological entropy, in general), and the sum is convergent for z sufficiently small, $|z| < 1/2M$. For large n the nth level sum (18.7) tends to the leading $\mathcal{L}^n$ eigenvalue e^{ns_0}. Summing this asymptotic estimate level by level

$$\Gamma(z) \approx \sum_{n=1}^{\infty} (ze^{s_0})^n = \frac{ze^{s_0}}{1 - ze^{s_0}} \tag{18.29}$$

we see that we should be able to determine s_0 by determining the smallest value of $z = e^{-s_0}$ for which the cycle expansion (18.28) diverges.

If one is interested only in the leading eigenvalue of $\mathcal{L}$, it suffices to consider the approximate trace $\Gamma(z)$. We will use this fact in sect. 19.3 to motivate the introduction of dynamical zeta functions (19.14), and in sect. 19.5 we shall give the exact relation between the exact and the approximate trace formulas.

Résumé

The description of a chaotic dynamical system in terms of cycles can be visualized as a tessellation of the dynamical system, figure 18.1, with a smooth flow approximated by its *periodic orbit skeleton*, each region $\mathcal{M}_i$ centered on a periodic point x_i of the topological length n, and the size of the region determined by the linearization of the flow around the periodic point. The integral over such topologically partitioned state space yields the *classical trace formula*

$$\sum_{\alpha=0}^{\infty} \frac{1}{s - s_\alpha} = \sum_p T_p \sum_{r=1}^{\infty} \frac{e^{r(\beta \cdot A_p - sT_p)}}{\left|\det\left(\mathbf{1} - M_p^r\right)\right|} \,.$$

Now that we have a trace formula, we might ask for what is it good? As it stands, it is little more than a scary divergent formula which relates the unspeakable infinity

of global eigenvalues to the unthinkable infinity of local unstable cycles. However, it is a good stepping stone on the way to construction of spectral determinants (to which we turn next), and a first hint that when the going is good, the theory might turn out to be convergent beyond our wildest dreams (chapter 23). In order to implement such formulas, we will have to determine "all" prime cycles. The first step is topological: enumeration of all admissible cycles undertaken in chapter 12. The more onerous enterprize of actually computing the cycles we first approach traditionally, as a numerical task in chapter 13, and then more boldly as a part and parcel of variational foundations of classical and quantum dynamics in chapter 29.

Commentary

Remark 18.1 Who's dunne it? Continuous time flow traces weighted by cycle periods were introduced by Bowen [18.1] who treated them as Poincaré section suspensions weighted by the "time ceiling" function (3.5). They were used by Parry and Pollicott [18.2].

Remark 18.2 Flat and sharp traces. In the above formal derivation of trace formulas we cared very little whether our sums were well posed. In the Fredholm theory traces like (18.14) require compact operators with continuous function kernels. This is not the case for our Dirac delta evolution operators: nevertheless, there is a large class of dynamical systems for which our results may be shown to be perfectly legal. In the mathematical literature expressions like (18.7) are called *flat* traces (see the review [18.4] and chapter 23). Other names for traces appear as well: for instance, in the context of 1 − *dimensional* mappings, *sharp* traces refer to generalizations of (18.7) where contributions of periodic points are weighted by the Lefschetz sign ±1, reflecting whether the periodic point sits on a branch of nth iterate of the map which crosses the diagonal starting from below or starting from above [19.10]. Such traces are connected to the theory of kneading invariants (see ref. [18.4] and references therein). Traces weighted by ±1 sign of the derivative of the fixed point have been used to study the period doubling repeller, leading to high precision estimates of the Feigenbaum constant δ, refs. [18.5, 20.6, 18.6].

Exercises

18.1. $t \to 0_+$ **regularization of eigenvalue sums****. In taking the Laplace transform (18.23) we have ignored the $t \to 0_+$ divergence, as we do not know how to regularize the delta function kernel in this limit. In the quantum (or heat kernel) case this limit gives rise to the Weyl or Thomas-Fermi mean eigenvalue spacing.Regularize the divergent sum in (18.23) and assign to such volume term some interesting role in the theory of classical resonance spectra. E-mail the solution to the authors.