Chapter 28

Noise

> He who establishes his argument by noise and command shows that his reason is weak.
>
> —M. de Montaigne

(G. Vattay and P. Cvitanović)

This chapter (which reader can safely skip on the first reading) is about noise, how it affects classical dynamics, and the ways it mimics quantum dynamics.

Why - in a monograph on deterministic and quantum chaos - start discussing noise? First, in physical settings any dynamics takes place against a noisy background, and whatever prediction we might have, we have to check its robustness to noise. Second, as we show in this chapter, to the leading order in noise strength, the semiclassical Hamilton-Jacobi formalism applies to weakly stochastic flows in toto. As classical noisy dynamics is more intuitive than quantum dynamics, understanding effects of noise helps demystify some of the formal machinery of semiclassical quantization. Surprisingly, symplectic structure emerges here not as a deep principle of mechanics, but an artifact of the leading approximation to quantum/noisy dynamics, not respected by higher order corrections. The same is true of semiclassical quantum dynamics; higher corrections do not respect canonical invariance. Third, the variational principle derived here turns out to be a powerful tool for determining periodic orbits, see chapter 29. And, last but not least, upon some reflection, the whole enterprize of replacing deterministic trajectories by deterministic evolution operators, chapters 16 to 20, seems fatally flowed; if we have given up infinite precision in specifying initial conditions, why do we alow ourselves the infinite precision in the specification of evolution laws, i.e., define the evolution operator by means of the Dirac delta function $\delta(y - f^t(x))$? It will be comforting to learn that the deterministic evolution operators survive unscathed, as the leading approximation to the noisy ones in the limit of weak noise.

We start by deriving the continuity equation for purely deterministic, noiseless

flow, and then incorporate noise in stages: diffusion equation, Langevin equation, Fokker-Planck equation, Hamilton-Jacobi formulation, stochastic path integrals.

28.1 Deterministic transport

(E.A. Spiegel and P. Cvitanović)

The large body of accrued wisdom on the subject of flows called fluid dynamics is about physical flows of media with continuous densities. On the other hand, the flows in state spaces of dynamical systems frequently require more abstract tools. To sharpen our intuition about those, it is helpful to outline the more tangible fluid dynamical vision.

Consider first the simplest property of a fluid flow called *material invariant*. A material invariant $I(x)$ is a property attached to each point x that is preserved by the flow, $I(x) = I(f^t(x))$; for example, at point $x(t) = f^t(x))$ a green particle (more formally: a *passive scalar*) is embedded into the fluid. As $I(x)$ is invariant, its total time derivative vanishes, $\dot{I}(x) = 0$. Written in terms of partial derivatives this is the *conservation equation* for the material invariant

$$\partial_t I + v \cdot \partial I = 0\,. \tag{28.1}$$

Let the *density* of representative points be $\rho(x,t)$. The manner in which the flow redistributes $I(x)$ is governed by a partial differential equation whose form is relatively simple because the representative points are neither created nor destroyed. This conservation property is expressed in the integral statement

$$\partial_t \int_V dx\, \rho I = - \int_{\partial V} d\sigma\, \hat{n}_i v_i \rho I\,,$$

where V is an arbitrary volume in the state space $\mathcal{M}$, ∂V is its surface, $\hat{n}$ is its outward normal, and repeated indices are summed over throughout. The divergence theorem turns the surface integral into a volume integral,

$$\int_V \left[\partial_t(\rho I) + \partial_i(v_i \rho I)\right] dx = 0\,,$$

where ∂_i is the partial derivative operator with respect to x_i. Since the integration is over an arbitrary volume, we conclude that

$$\partial_t(\rho I) + \partial_i(\rho I v_i) = 0\,. \tag{28.2}$$

The choice $I \equiv 1$ yields the *continuity equation* for the density:

$$\partial_t \rho + \partial_i(\rho v_i) = 0\,. \tag{28.3}$$

We have used here the language of fluid mechanics to ease the visualization, but, as we already saw in (16.25), our previous derivation of the continuity equation, any deterministic state space flow satisfies the continuity equation.

28.2 Brownian diffusion

Consider tracer molecules, let us say big, laggardly green molecules, embedded in a denser gas of light molecules. Assume that the density of tracer molecules ρ compared to the background gas density is low, so we can neglect green-green collisions. Each green molecule, jostled by frequent collisions with the background gas, executes its own Brownian motion. The molecules are neither created nor destroyed, so their number within an arbitrary volume V changes with time only by the current density j_i flow through its surface ∂V (with $\hat{n}$ its outward normal):

$$\partial_t \int_V dx\,\rho = -\int_{\partial V} d\sigma\,\hat{n}_i\, j_i\,. \tag{28.4}$$

The divergence theorem turns this into the conservation law for tracer density:

$$\partial_t \rho + \partial_i j_i = 0\,. \tag{28.5}$$

The tracer density ρ is defined as the average density of a ‘material particle,’ averaged over a subvolume large enough to contain many green (and still many more background) molecules, but small compared to the macroscopic observational scales. What is j? If the density is constant, on the average as many molecules leave the material particle volume as they enter it, so a reasonable phenomenological assumption is that the *average* current density (*not* the individual particle current density ρv_i in (28.3)) is driven by the density gradient

$$j_i = -D\frac{\partial\rho}{\partial x_i}\,. \tag{28.6}$$

This is the *Fick law*, with the diffusion constant D a phenomenological parameter. For simplicity here we assume that D is a scalar; in general $D \to D_{ij}(x,t)$ is a space- and time-dependent tensor. Substituting this j into (28.5) yields the *diffusion equation*

$$\frac{\partial}{\partial t}\rho(x,t) = D\frac{\partial^2}{\partial x^2}\rho(x,t)\,. \tag{28.7}$$

This linear equation has an exact solution in terms of an initial Dirac delta density distribution, $\rho(x,0) = \delta(x-x_0)$,

$$\rho(x,t) = \frac{1}{(4\pi Dt)^{d/2}} e^{-\frac{(x-x_0)^2}{4Dt}} = \frac{1}{(4\pi Dt)^{d/2}} e^{-\frac{\dot{x}^2}{4Dt}}\,. \tag{28.8}$$

The average distance covered in time t obeys the Einstein diffusion formula

$$\left\langle (x-x_0)^2 \right\rangle_t = \int dx\,\rho(x,t)(x-x_0)^2 = 2dDt\,. \tag{28.9}$$

28.3 Weak noise

> The connection between path integration and Brownian motion is so close that they are nearly indistinguishable. Unfortunately though, like a body and its mirror image, the sum over paths for Brownian motion is a theory having substance, while its path integral image exists mainly in the eye of the beholder.
>
> —L. S. Schulman

So far we have considered tracer molecule dynamics which is purely Brownian, with no deterministic “drift.” Consider next a deterministic flow $\dot{x} = v(x)$ perturbed by a stochastic term $\xi(t)$,

$$\dot{x} = v(x) + \xi(t)\,. \tag{28.10}$$

We shall refer to equations of this type as *Langevin equations*. Assume that $\xi(t)$’s fluctuate around $[\dot{x} - v(x)]$ with a Gaussian probability density

$$P(\xi,\delta t) = \left(\frac{\delta t}{4\pi D}\right)^{d/2} e^{-\frac{\xi^2}{4D}\delta t}\,, \tag{28.11}$$

and are uncorrelated in time (white noise)

$$\langle \xi(t)\xi(t')\rangle = 2dD\delta(t-t')\,. \tag{28.12}$$

The normalization factors in (28.8) and (28.11) differ, as $p(\xi,\delta t)$ is a probability density for velocity ξ, and $\rho(x,t)$ is a probability density for position x. The material particle now drifts along the trajectory $x(t)$, so the velocity diffusion follows (28.8) for infinitesimal time δt only. As $D \to 0$, the distribution tends to the (noiseless, deterministic) Dirac delta function.

The phenomenological Fick law current (28.6) is now a sum of two components, the material particle center-of-mass deterministic drift $v(x)$ and the weak noise term

$$j_i = \rho v_i - D\frac{\partial\rho}{\partial x_i}\,, \tag{28.13}$$

Substituting this j into (28.5) yields the *Fokker-Planck equation*

$$\partial_t \rho + \partial_i(\rho v_i) = D\,\partial^2 \rho . \tag{28.14}$$

The left hand side, $d\rho/dt = \partial_t \rho + \partial \cdot (\rho v)$, is deterministic, with the continuity equation (28.3) recovered in the weak noise limit $D \to 0$. The right hand side describes the diffusive transport in or out of the material particle volume. If the density is lower than in the immediate neighborhood, the local curvature is positive, $\partial^2 \rho > 0$, and the density grows. Conversely, for negative curvature diffusion lowers the local density, thus smoothing the variability of ρ. Where is the density going globally?

If the system is bound, the probability density vanishes sufficiently fast outside the central region, $\rho(x,t) \to 0$ as $|x| \to \infty$, and the total probability is conserved

$$\int dx\, \rho(x,t) = 1 .$$

Any initial density $\rho(x,0)$ is smoothed by diffusion and with time tends to the invariant density

$$\rho_0(x) = \lim_{t\to\infty} \rho(x,t) , \tag{28.15}$$

an eigenfunction $\rho(x,t) = e^{st}\rho_0(x)$ of the time-independent Fokker-Planck equation

$$\left(\partial_i v_i - D\,\partial^2 + s_\alpha\right)\rho_\alpha = 0 , \tag{28.16}$$

with vanishing eigenvalue $s_0 = 0$. Provided the noiseless classical flow is hyperbolic, in the vanishing noise limit the leading eigenfunction of the Fokker-Planck equation tends to natural measure (16.17) of the corresponding deterministic flow, the leading eigenvector of the Perron-Frobenius operator.

If the system is open, there is a continuous outflow of probability from the region under study, the leading eigenvalue is contracting, $s_0 < 0$, and the density of the system tends to zero. In this case the leading eigenvalue s_0 of the time-independent Fokker-Planck equation (28.16) can be interpreted by saying that a finite density can be maintained by pumping back probability into the system at a constant rate $\gamma = -s_0$. The value of γ for which any initial probability density converges to a finite equilibrium density is called the *escape rate*. In the noiseless limit this coincides with the deterministic escape rate (17.15).

We have introduced noise phenomenologically, and used the weak noise assumption in retaining only the first derivative of ρ in formulating the Fick law (28.6) and including noise additively in (28.13). A full theory of stochastic ODEs is much subtler, but this will do for our purposes.

28.4 Weak noise approximation

In the spirit of the WKB approximation, we shall now study the evolution of the probability distribution by rewriting it as

$$\rho(x,t) = e^{\frac{1}{2D}R(x,t)} . \tag{28.17}$$

The time evolution of R is given by

$$\partial_t R + v\partial R + (\partial R)^2 = D\partial v + D\partial^2 R .$$

Consider now the weak noise limit and drop the terms proportional to D. The remaining equation

$$\partial_t R + H(x, \partial R) = 0$$

is known as the Hamilton-Jacobi equation . The function R can be interpreted as the Hamilton's principal function, corresponding to the Hamiltonian

$$H(x,p) = p\,v(x) + p^2/2 ,$$

with the Hamilton's equations of motion

$$\begin{aligned} \dot{x} &= \partial_p H = v + p \\ \dot{p} &= -\partial_x H = -A^T p , \end{aligned} \tag{28.18}$$

where A is the stability matrix (4.3)

$$A_{ij}(x) = \frac{\partial v_i(x)}{\partial x_j} .$$

The noise Lagrangian is then

$$L(x,\dot{x}) = \dot{x}\cdot p - H = \frac{1}{2}\left[\dot{x} - v(x)\right]^2 . \tag{28.19}$$

We have come the full circle - the Lagrangian is the exponent of our assumed Gaussian distribution (28.11) for noise $\xi^2 = [\dot{x} - v(x)]^2$. What is the meaning of this Hamiltonian, Lagrangian? Consider two points x_0 and x. Which noisy path is the most probable path that connects them in time t? The probability of a given path $\mathcal{P}$ is given by the probability of the noise sequence $\xi(t)$ which generates the path. This probability is proportional to the product of the noise probability functions (28.11) along the path, and the total probability for reaching x from x_0 in time t is given by the sum over all paths, or the stochastic path integral (Wiener integral)

$$
\begin{aligned}
P(x, x_0, t) \quad &\sim \quad \sum_{\mathcal{P}} \prod_j p(\xi(\tau_j), \delta\tau_j) = \int \prod_j d\xi_j \left(\frac{\delta\tau_j}{4\pi D}\right)^{d/2} e^{-\frac{\xi(\tau_j)^2}{4D}\delta\tau_i} \\
&\to \quad \frac{1}{Z} \sum_{\mathcal{P}} \exp\left(-\frac{1}{4D}\int_0^t d\tau\, \xi^2(\tau)\right), \qquad (28.20)
\end{aligned}
$$

where $\delta\tau_i = \tau_i - \tau_i$, and the normalization constant is

$$
\frac{1}{Z} = \lim \prod_i \left(\frac{\delta\tau_i}{2\pi D}\right)^{d/2} .
$$

The most probable path is the one maximizing the integral inside the exponential. If we express the noise (28.10) as

$$
\xi(t) = \dot{x}(t) - v(x(t)) ,
$$

the probability is maximized by the variational principle

$$
\min \int_0^t d\tau [\dot{x}(\tau) - v(x(\tau))]^2 = \min \int_0^t L(x(\tau), \dot{x}(\tau)) d\tau .
$$

By the standard arguments, for a given x, x' and t the probability is maximized by a solution of Hamilton's equations (28.18) that connects the two points $x_0 \to x'$ in time t.

Résumé

When a deterministic trajectory is smeared out under the influence of Gaussian noise of strength D, the deterministic dynamics is recovered in the weak noise limit $D \to 0$. The effect of the noise can be taken into account by adding noise corrections to the classical trace formula.

Commentary

Remark 28.1 Literature. The theory of stochastic processes is a vast subject, spanning over centuries and over disciplines ranging from pure mathematics to impure finance.

We enjoyed reading van Kampen classic [28.1], especially his railings against those who blunder carelessly into nonlinear landscapes. Having committed this careless chapter to print, we shall no doubt be cast to a special place on the long list of van Kampen's sinners (and not for the first time, either). A more specialized monograph like Risken's [28.2] will do just as well. The standard Langevin equation is a stochastic equation for a Brownian particle, in which one replaces the Newton's equation for force by two counter-balancing forces: random accelerations $\xi(t)$ which tend to smear out a particle trajectory, and a damping term which drives the velocity to zero. Here we denote by 'Langevin equation' a more general family of stochastic differential equations (28.10) with additive weak noise limit.

If a flow is linear (in Hamiltonian case, with harmonic oscillator potential) with an attractive fixed point, $\mathcal{L}^t_D$ describes a version of the Ornstein-Uhlenbeck process [28.20], (introduced already by Laplace in 1810, see ref. [28.21]). Gaussians are often rediscovered, so Onsager-Machlup seminal paper [28.18], which studies the same attractive linear fixed point is in literature often credited for being the first to introduce a variational method - the "principle of least dissipation" - based on the Lagrangian of form (28.19). They, in turn, credit Rayleigh [28.19] with introducing the least dissipation principle in hydrodynamics. Onsager-Machlup paper deals only with a finite set of linearly damped thermodynamic variables, and not with a nonlinear flow or unstable periodic orbits. In our exposition the setting is much more general: we study fluctuations over a state space varying velocity field $v(x)$. Schulman's monograph [28.11] contains a very readable summary of Kac's [28.12] exposition of Wiener's integral over stochastic paths.

Exercises

28.1. **Who ordered $\sqrt{\pi}$?** Derive the Gaussian integral

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-\frac{x^2}{2a}} = \sqrt{a}\,, \qquad a > 0\,.$$

assuming only that you know to integrate the exponential function e^{-x}. Hint, hint: x^2 is a radius-squared of something. π is related to the area or circumference of something.

28.2. ***D*-dimensional Gaussian integrals.** Show that the Gaussian integral in D-dimensions is given by

$$\frac{1}{(2\pi)^{d/2}} \int d^d\phi e^{-\frac{1}{2}\phi^T \cdot M^{-1} \cdot \phi + \phi \cdot J} = |\det M|^{\frac{1}{2}}\, e^{\frac{1}{2} J^T \cdot M \cdot J} \tag{28.21}$$

where M is a real positive definite $[d \times d]$ matrix, i.e., a matrix with strictly positive eigenvalues. x, J are D-dimensional vectors, and x^T is the transpose of x.

28.3. **Convolution of Gaussians.** Show that the Fourier transform of convolution

$$[f * g](x) = \int d^d y\, f(x-y) g(y)$$

of two Gaussians

$$f(x) = e^{-\frac{1}{2} x^T \cdot \frac{1}{\Delta_1} \cdot x}\,, \qquad g(x) = e^{-\frac{1}{2} x^T \cdot \frac{1}{\Delta_2} \cdot x}$$

factorizes as

$$[f * g](x) = \frac{1}{(2\pi)^d} \int dk\, F(k) G(k) e^{ik \cdot x}\,, \tag{28.22}$$

where

$$\begin{aligned} F(k) &= \frac{1}{(2\pi)^d} \int d^d x\, f(x) e^{-ik \cdot x} = |\det \Delta_1|^{1/2} e^{\frac{1}{2} k^T \cdot \Delta_1 \cdot k} \\ G(k) &= \frac{1}{(2\pi)^d} \int d^d x\, g(x) e^{-ik \cdot x} = |\det \Delta_2|^{1/2} e^{\frac{1}{2} k^T \cdot \Delta_2 \cdot k} \end{aligned}$$

Hence

$$\begin{aligned} [f * g](x) &= \frac{1}{(2\pi)^d} |\det \Delta_1 \det \Delta_1|^{1/2} \int d^d p\, e^{\frac{1}{2} p^T \cdot (\Delta_1 + \Delta} \\ &= \left| \frac{\det \Delta_1 \det \Delta_2}{\det(\Delta_1 + \Delta_2)} \right|^{1/2} e^{-\frac{1}{2} x^T \cdot (\Delta_1 + \Delta_2)^{-1} \cdot x}\,. \qquad (\end{aligned}$$

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