# Chapter 19

# Spectral determinants

> "It seems very pretty," she said when she had finished it, "but it's rather hard to understand!" (You see she didn't like to confess, even to herself, that she couldn't make it out at all.) "Somehow it seems to fill my head with ideas — only I don't exactly know what they are!"
>
> —Lewis Carroll, *Through the Looking Glass*

THE PROBLEM with the trace formulas (18.10), (18.23) and (18.28) is that they diverge at $z = e^{-s_0}$, respectively $s = s_0$, i.e., precisely where one would like to use them. While this does not prevent numerical estimation of some "thermodynamic" averages for iterated mappings, in the case of the Gutzwiller trace formula this leads to a perplexing observation that crude estimates of the radius of convergence seem to put the entire physical spectrum out of reach. We shall now cure this problem by thinking, at no extra computational cost; while traces and determinants are formally equivalent, determinants are the tool of choice when it comes to computing spectra. Determinants tend to have larger analyticity domains because if $\mathrm{tr}\,\mathcal{L}/(1 - z\mathcal{L}) = -\frac{d}{dz} \ln \det(1 - z\mathcal{L})$ diverges at a particular value of $z$, then $\det(1 - z\mathcal{L})$ might have an isolated zero there, and a zero of a function is easier to determine numerically than its poles. chapter 23

## 19.1 Spectral determinants for maps

The eigenvalues $z_k$ of a linear operator are given by the zeros of the determinant

$$\det(1 - z\mathcal{L}) = \prod_k (1 - z/z_k) \,. \qquad (19.1)$$

For finite matrices this is the characteristic determinant; for operators this is the Hadamard representation of the *spectral determinant* (sparing the reader from pondering possible regularization factors). Consider first the case of maps, for which the evolution operator advances the densities by integer steps in time. In this case we can use the formal matrix identity exercise 4.1

$$\ln \det (1 - M) = \operatorname{tr} \ln(1 - M) = -\sum_{n=1}^{\infty} \frac{1}{n} \operatorname{tr} M^n \,, \tag{19.2}$$

to relate the spectral determinant of an evolution operator for a map to its traces (18.8), and hence to periodic orbits:

$$\begin{aligned} \det (1 - z\mathcal{L}) &= \exp\left( -\sum_{n}^{\infty} \frac{z^n}{n} \operatorname{tr} \mathcal{L}^n \right) \\ &= \exp\left( -\sum_{p} \sum_{r=1}^{\infty} \frac{1}{r} \frac{z^{n_p r} e^{r\beta \cdot A_p}}{\left| \det \left( \mathbf{1} - M_p^r \right) \right|} \right) . \end{aligned} \tag{19.3}$$

Going the other way, the trace formula (18.10) can be recovered from the spectral determinant by taking a derivative

$$\operatorname{tr} \frac{z\mathcal{L}}{1 - z\mathcal{L}} = -z \frac{d}{dz} \ln \det (1 - z\mathcal{L}) \,. \tag{19.4}$$

fast track:
sect. 19.2, p. 364

**Example 19.1** ***Spectral determinants of transfer operators:***

*For a piecewise-linear map (17.17) with a finite Markov partition, an explicit formula for the spectral determinant follows by substituting the trace formula (18.11) into (19.3):*

$$\det (1 - z\mathcal{L}) = \prod_{k=0}^{\infty} \left( 1 - \frac{t_0}{\Lambda_0^k} - \frac{t_1}{\Lambda_1^k} \right) \,, \tag{19.5}$$

*where $t_s = z/|\Lambda_s|$. The eigenvalues are necessarily the same as in (18.12), which we already determined from the trace formula (18.10).*

*The exponential spacing of eigenvalues guarantees that the spectral determinant (19.5) is an entire function. It is this property that generalizes to piecewise smooth flows with finite Markov partitions, and singles out spectral determinants rather than the trace formulas or dynamical zeta functions as the tool of choice for evaluation of spectra.*

## 19.2 Spectral determinant for flows

... an analogue of the [Artin-Mazur] zeta function for diffeomorphisms seems quite remote for flows. However we will mention a wild idea in this direction. $[\cdots]$ define $l(\gamma)$ to be the minimal period of $\gamma$ $[\cdots]$ then define formally (another zeta function!) $Z(s)$ to be the infinite product

$$Z(s) = \prod_{\gamma \in \Gamma} \prod_{k=0}^{\infty} \left(1 - [\exp l(\gamma)]^{-s-k}\right) .$$

—Stephen Smale, *Differentiable Dynamical Systems*

We write the formula for the spectral determinant for flows by analogy to (19.3)

$$\det(s - \mathcal{A}) = \exp\left(-\sum_p \sum_{r=1}^{\infty} \frac{1}{r} \frac{e^{r(\beta \cdot A_p - sT_p)}}{\left|\det\left(\mathbf{1} - M_p^r\right)\right|}\right) , \tag{19.6}$$

and then check that the trace formula (18.23) is the logarithmic derivative of the spectral determinant

$$\mathrm{tr}\, \frac{1}{s - \mathcal{A}} = \frac{d}{ds} \ln \det(s - \mathcal{A}) . \tag{19.7}$$

With $z$ set to $z = e^{-s}$ as in (18.24), the spectral determinant (19.6) has the same form for both maps and flows. We refer to (19.6) as *spectral determinant*, as the spectrum of the operator $\mathcal{A}$ is given by the zeros of

$$\det(s - \mathcal{A}) = 0 . \tag{19.8}$$

We now note that the $r$ sum in (19.6) is close in form to the expansion of a logarithm. This observation enables us to recast the spectral determinant into an infinite product over periodic orbits as follows:

Let $M_p$ be the $p$-cycle $[d \times d]$ transverse Jacobian matrix, with eigenvalues $\Lambda_{p,1}, \Lambda_{p,2}, \ldots, \Lambda_{p,d}$. Expanding the expanding eigenvalue factors $1/(1 - 1/\Lambda_{p,e})$ and the contracting eigenvalue factors $1/(1 - \Lambda_{p,c})$ in (18.4) as geometric series, substituting back into (19.6), and resumming the logarithms, we find that the spectral determinant is formally given by the infinite product

$$\det(s - \mathcal{A}) = \prod_{k_1=0}^{\infty} \cdots \prod_{l_c=0}^{\infty} \frac{1}{\zeta_{k_1 \cdots l_c}}$$

$$1/\zeta_{k_1 \cdots l_c} = \prod_p \left(1 - t_p \frac{\Lambda_{p,e+1}^{l_1} \Lambda_{p,e+2}^{l_2} \cdots \Lambda_{p,d}^{l_c}}{\Lambda_{p,1}^{k_1} \Lambda_{p,2}^{k_2} \cdots \Lambda_{p,e}^{k_e}}\right) \tag{19.9}$$

$$t_p = t_p(z, s, \beta) = \frac{1}{|\Lambda_p|} e^{\beta \cdot A_p - sT_p} z^{n_p} . \tag{19.10}$$

In such formulas $t_p$ is a weight associated with the $p$ cycle (letter $t$ refers to the "local trace" evaluated along the $p$ cycle trajectory), and the index $p$ runs through all distinct prime cycles. Why the factor $z^{n_p}$? It is associated with the trace formula (18.10) for maps, whereas the factor $e^{-sT_p}$ is specific to the continuous time trace formuls (18.23); according to (18.24) we should use either one or the other. But we have learned in sect. 3.1 that flows can be represented either by their continuous-time trajectories, or by their topological time Poincaré section return maps. In cases when we have good control over the topology of the flow, it is often convenient to insert the $z^{n_p}$ factor into cycle weights, as a formal parameter which keeps track of the topological cycle lengths. These factors will assist us in expanding zeta functions and determinants, eventually we shall set $z = 1$. The subscripts $e, c$ indicate that there are $e$ expanding eigenvalues, and $c$ contracting eigenvalues. The observable whose average we wish to compute contributes through the $A^t(x)$ term in the $p$ cycle multiplicative weight $e^{\beta \cdot A_p}$. By its definition (17.1), the weight for maps is a product along the periodic points

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$$e^{A_p} = \prod_{j=0}^{n_p-1} e^{a(f^j(x_p))} ,$$

and the weight for flows is an exponential of the integral (17.5) along the cycle

$$e^{A_p} = \exp\left(\int_0^{T_p} a(x(\tau))d\tau\right).$$

This formula is correct for scalar weighting functions; more general matrix valued weights require a time-ordering prescription as in the Jacobian matrix of sect. 4.1.

**Example 19.2** ***Expanding*** $1-dimensional$ ***map:*** *For expanding* $1-dimensional$ *mappings the spectral determinant (19.9) takes the form*

$$\det(1 - z\mathcal{L}) = \prod_p \prod_{k=0}^{\infty} \left(1 - t_p/\Lambda_p^k\right) , \qquad t_p = \frac{e^{\beta A_p}}{|\Lambda_p|} z^{n_p} . \tag{19.11}$$

**Example 19.3** ***Two-degree of freedom Hamiltonian flows:*** *For a 2-degree of freedom Hamiltonian flows the energy conservation eliminates on phase space variable, and restriction to a Poincaré section eliminates the marginal longitudinal eigenvalue* $\Lambda = 1$*, so a periodic orbit of 2-degree of freedom hyperbolic Hamiltonian flow has one expanding transverse eigenvalue* $\Lambda$, $|\Lambda| > 1$, *and one contracting transverse eigenvalue* $1/\Lambda$. *The weight in (18.4) is expanded as follows:*

$$\frac{1}{\left|\det\left(\mathbf{1} - M_p^r\right)\right|} = \frac{1}{|\Lambda|^r (1 - 1/\Lambda_p^r)^2} = \frac{1}{|\Lambda|^r} \sum_{k=0}^{\infty} \frac{k+1}{\Lambda_p^{kr}} . \tag{19.12}$$

*The spectral determinant exponent can be resummed,*

$$-\sum_{r=1}^{\infty} \frac{1}{r} \frac{e^{(\beta A_p - sT_p)r}}{\left|\det\left(\mathbf{1} - M_p^r\right)\right|} = \sum_{k=0}^{\infty} (k+1) \log\left(1 - \frac{e^{\beta A_p - sT_p}}{|\Lambda_p|\Lambda_p^k}\right) ,$$

*and the spectral determinant for a 2-dimensional hyperbolic Hamiltonian flow rewritten as an infinite product over prime cycles*

$$\det(s - \mathcal{A}) = \prod_p \prod_{k=0}^{\infty} \left(1 - t_p/\Lambda_p^k\right)^{k+1} . \tag{19.13}$$

exercise 23.4

Now we are finally poised to deal with the problem posed at the beginning of chapter 18; how do we actually evaluate the averages introduced in sect. 17.1? The eigenvalues of the dynamical averaging evolution operator are given by the values of $s$ for which the spectral determinant (19.6) of the evolution operator (17.23) vanishes. If we can compute the leading eigenvalue $s_0(\beta)$ and its derivatives, we are done. Unfortunately, the infinite product formula (19.9) is no more than a shorthand notation for the periodic orbit weights contributing to the spectral determinant; more work will be needed to bring such formulas into a tractable form. This shall be accomplished in chapter 20, but here it is natural to introduce still another variant of a determinant, the dynamical zeta function.

## 19.3 Dynamical zeta functions

It follows from sect. 18.1.1 that if one is interested only in the leading eigenvalue of $\mathcal{L}^t$, the size of the $p$ cycle neighborhood can be approximated by $1/|\Lambda_p|^r$, the dominant term in the $rT_p = t \to \infty$ limit, where $\Lambda_p = \prod_e \Lambda_{p,e}$ is the product of the expanding eigenvalues of the Jacobian matrix $M_p$. With this replacement the spectral determinant (19.6) is replaced by the *dynamical zeta function*

$$1/\zeta = \exp\left(-\sum_p \sum_{r=1}^{\infty} \frac{1}{r} t_p^r\right) \tag{19.14}$$

that we have already derived heuristically in sect. 1.5.2. Resumming the logarithms using $\sum_r t_p^r/r = -\ln(1-t_p)$ we obtain the *Euler product representation* of the dynamical zeta function:

$$1/\zeta = \prod_p \left(1 - t_p\right) . \tag{19.15}$$

In order to simplify the notation, we usually omit the explicit dependence of $1/\zeta$, $t_p$ on $z$, $s$, $\beta$ whenever the dependence is clear from the context.

The approximate trace formula (18.28) plays the same role *vis-à-vis* the dynamical zeta function (19.7)

$$\Gamma(s) = \frac{d}{ds} \ln \zeta^{-1} = \sum_p \frac{T_p t_p}{1 - t_p}\,, \tag{19.16}$$

as the exact trace formula (18.23) plays *vis-à-vis* the spectral determinant (19.6). The heuristically derived dynamical zeta function of sect. 1.5.2 now re-emerges as the $1/\zeta_{0\cdots 0}(z)$ part of the *exact* spectral determinant; other factors in the infinite product (19.9) affect the non-leading eigenvalues of $\mathcal{L}$.

In summary, the dynamical zeta function (19.15) associated with the flow $f^t(x)$ is defined as the product over all prime cycles $p$. The quantities, $T_p$, $n_p$ and $\Lambda_p$, denote the period, topological length and product of the expanding Floquet multipliers of prime cycle $p$, $A_p$ is the integrated observable $a(x)$ evaluated on a single traversal of cycle $p$ (see (17.5)), $s$ is a variable dual to the time $t$, $z$ is dual to the discrete “topological” time $n$, and $t_p(z, s, \beta)$ denotes the local trace over the cycle $p$. We have included the factor $z^{n_p}$ in the definition of the cycle weight in order to keep track of the number of times a cycle traverses the surface of section. The dynamical zeta function is useful because the term

$$1/\zeta(s) = 0 \tag{19.17}$$

when $s = s_0$, Here $s_0$ is the leading eigenvalue of $\mathcal{L}^t = e^{t\mathcal{A}}$, which is often all that is necessary for application of this equation. The above argument completes our derivation of the trace and determinant formulas for classical chaotic flows. In chapters that follow we shall make these formulas tangible by working out a series of simple examples.

The remainder of this chapter offers examples of zeta functions.

fast track:
chapter 20, p. 378

### 19.3.1 A contour integral formulation

The following observation is sometimes useful, in particular for zeta functions with richer analytic structure than just zeros and poles, as in the case of intermittency (chapter 24): $\Gamma_n$, the trace sum (18.26), can be expressed in terms of the dynamical zeta function (19.15)

$$1/\zeta(z) = \prod_p \left(1 - \frac{z^{n_p}}{|\Lambda_p|}\right) . \tag{19.18}$$

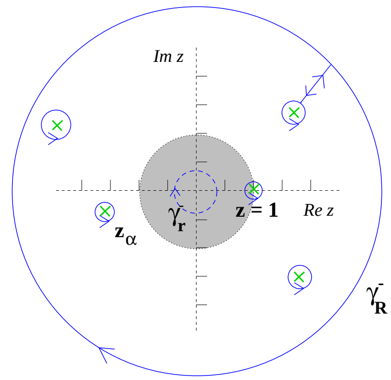


**Figure 19.1:** The survival probability $\Gamma_n$ can be split into contributions from poles (x) and zeros (o) between the small and the large circle and a contribution from the large circle.

as a contour integral

$$\Gamma_n = \frac{1}{2\pi i}\oint_{\gamma_r^-} z^{-n}\left(\frac{d}{dz}\log\zeta^{-1}(z)\right)dz \ , \tag{19.19}$$

exercise 19.7

where a small contour $\gamma_r^-$ encircles the origin in negative (clockwise) direction. If the contour is small enough, i.e., it lies inside the unit circle $|z| = 1$, we may write the logarithmic derivative of $\zeta^{-1}(z)$ as a convergent sum over all periodic orbits. Integrals and sums can be interchanged, the integrals can be solved term by term, and the trace formula (18.26) is recovered. For hyperbolic maps, cycle expansions or other techniques provide an analytical continuation of the dynamical zeta function beyond the leading zero; we may therefore deform the original contour into a larger circle with radius $R$ which encircles both poles and zeros of $\zeta^{-1}(z)$, as depicted in figure 19.1. Residue calculus turns this into a sum over the zeros $z_\alpha$ and poles $z_\beta$ of the dynamical zeta function, that is

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$$\Gamma_n = \sum_{|z_\alpha|<R}^{\text{zeros}} \frac{1}{z_\alpha^n} - \sum_{|z_\beta|<R}^{\text{poles}} \frac{1}{z_\beta^n} + \frac{1}{2\pi i}\oint_{\gamma_R^-} dz\, z^{-n}\frac{d}{dz}\log\zeta^{-1}, \tag{19.20}$$

where the last term gives a contribution from a large circle $\gamma_R^-$. It would be a miracle if you still remembered this, but in sect. 1.4.3 we interpreted $\Gamma_n$ as fraction of survivors after $n$ bounces, and defined the escape rate $\gamma$ as the rate of the find exponential decay of $\Gamma_n$. We now see that this exponential decay is dominated by the leading zero or pole of $\zeta^{-1}(z)$.

### 19.3.2 Dynamical zeta functions for transfer operators

Ruelle's original dynamical zeta function was a generalization of the topological zeta function (15.27) to a function that assigns different weights to different cycles:

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$$\zeta(z) = \exp\sum_{n=1}^{\infty}\frac{z^n}{n}\left(\sum_{x_i\in \text{Fix} f^n} \text{tr}\prod_{j=0}^{n-1} g(f^j(x_i))\right) .$$

exercise 18.2

Here we sum over all periodic points $x_i$ of period $n$, and $g(x)$ is any (matrix valued) weighting function, where the weight evaluated multiplicatively along the trajectory of $x_i$.

By the chain rule (4.51) the stability of any $n$-cycle of a $1-$ *dimensional* map is given by $\Lambda_p = \prod_{j=1}^{n} f'(x_i)$, so the $1-$ *dimensional* map cycle stability is the simplest example of a multiplicative cycle weight $g(x_i) = 1/|f'(x_i)|$, and indeed - via the Perron-Frobenius evolution operator (16.9) - the historical motivation for Ruelle's more abstract construction.

In particular, for a piecewise-linear map with a finite Markov partition such as the map of example 16.1, the dynamical zeta function is given by a finite polynomial, a straightforward generalization of the topological transition matrix determinant (14.1). As explained in sect. 15.3, for a finite $[N \times N]$ dimensional matrix the determinant is given by

$$\prod_p (1 - t_p) = \sum_{n=1}^{N} z^n c_n \,,$$

where $c_n$ is given by the sum over all non-self-intersecting closed paths of length $n$ together with products of all non-intersecting closed paths of total length $n$.

**Example 19.4** ***A piecewise linear repeller:*** *Due to piecewise linearity, the stability of any* $n$*-cycle of the piecewise linear repeller (17.17) factorizes as* $\Lambda_{s_1 s_2 \ldots s_n} = \Lambda_0^m \Lambda_1^{n-m}$*, where* $m$ *is the total number of times the letter* $s_j = 0$ *appears in the* $p$ *symbol sequence, so the traces in the sum (18.28) take the particularly simple form*

$$\mathrm{tr}\, T^n = \Gamma_n = \left( \frac{1}{|\Lambda_0|} + \frac{1}{|\Lambda_1|} \right)^n \,.$$

*The dynamical zeta function (19.14) evaluated by resumming the traces,* exercise 19.3

$$1/\zeta(z) = 1 - z/|\Lambda_0| - z/|\Lambda_1| \,, \qquad (19.21)$$

*is indeed the determinant* $\det(1 - zT)$ *of the transfer operator (17.19), which is almost as simple as the topological zeta function (15.34).*

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More generally, piecewise-linear approximations to dynamical systems yield polynomial or rational polynomial cycle expansions, provided that the symbolic dynamics is a subshift of finite type.

We see that the exponential proliferation of cycles so dreaded by quantum chaologians is a bogus anxiety; we are dealing with exponentially many cycles of increasing length and instability, but all that really matters in this example are the stabilities of the two fixed points. Clearly the information carried by the infinity of longer cycles is highly redundant; we shall learn in chapter 20 how to exploit this redundancy systematically.

## 19.4 False zeros

Compare (19.21) with the Euler product (19.15). For simplicity consider two equal scales, $|\Lambda_0| = |\Lambda_1| = e^{\lambda}$. Our task is to determine the leading zero $z = e^{\gamma}$ of the Euler product. It is a novice error to assume that the infinite Euler product (19.15) vanishes whenever one of its factors vanishes. If that were true, each factor $(1 - z^{n_p}/|\Lambda_p|)$ would yield

$$0 = 1 - e^{n_p(\gamma - \lambda_p)}, \qquad (19.22)$$

so the escape rate $\gamma$ would equal the Floquet exponent of a repulsive cycle, one eigenvalue $\gamma = \gamma_p$ for each prime cycle $p$. This is false! The exponentially growing number of cycles with growing period conspires to shift the zeros of the infinite product. The correct formula follows from (19.21)

$$0 = 1 - e^{\gamma - \lambda + h}, \qquad h = \ln 2. \qquad (19.23)$$

This particular formula for the escape rate is a special case of a general relation between escape rates, Lyapunov exponents and entropies that is not yet included into this book. Physically this means that the escape induced by the repulsion by each unstable fixed point is diminished by the rate of backscatter from other repelling regions, i.e., the entropy $h$; the positive entropy of orbits shifts the "false zeros" $z = e^{\lambda_p}$ of the Euler product (19.15) to the true zero $z = e^{\lambda - h}$.

## 19.5 Spectral determinants *vs.* dynamical zeta functions

In sect. 19.3 we derived the dynamical zeta function as an approximation to the spectral determinant. Here we relate dynamical zeta functions to spectral determinants *exactly*, by showing that a dynamical zeta function can be expressed as a ratio of products of spectral determinants.

The elementary identity for $d$-dimensional matrices

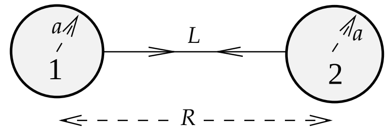
$$1 = \frac{1}{\det(1 - M)} \sum_{k=0}^{d} (-1)^k \operatorname{tr}\left(\wedge^k M\right), \qquad (19.24)$$

inserted into the exponential representation (19.14) of the dynamical zeta function, relates the dynamical zeta function to *weighted* spectral determinants.

**Example 19.5** ***Dynamical zeta function in terms of determinants,*** $1 - dimensional$ ***maps:*** *For* $1 - dimensional$ *maps the identity*

$$1 = \frac{1}{(1 - 1/\Lambda)} - \frac{1}{\Lambda}\frac{1}{(1 - 1/\Lambda)}$$

**Figure 19.2:** A game of pinball consisting of two disks of equal size in a plane, with its only periodic orbit (A. Wirzba).




*substituted into (19.14) yields an expression for the dynamical zeta function for* $1-dimensional$ *maps as a ratio of two spectral determinants*

$$1/\zeta = \frac{\det(1 - z\mathcal{L})}{\det(1 - z\mathcal{L}_{(1)})} \tag{19.25}$$

*where the cycle weight in* $\mathcal{L}_{(1)}$ *is given by replacement* $t_p \to t_p/\Lambda_p$*. As we shall see in chapter 23, this establishes that for nice hyperbolic flows* $1/\zeta$ *is meromorphic, with poles given by the zeros of* $\det(1 - z\mathcal{L}_{(1)})$*. The dynamical zeta function and the spectral determinant have the same zeros, although in exceptional circumstances some zeros of* $\det(1 - z\mathcal{L}_{(1)})$ *might be cancelled by coincident zeros of* $\det(1 - z\mathcal{L}_{(1)})$*. Hence even though we have derived the dynamical zeta function in sect. 19.3 as an "approximation" to the spectral determinant, the two contain the same spectral information.*

**Example 19.6** ***Dynamical zeta function in terms of determinants,*** $2-dimensional$ ***Hamiltonian maps:*** *For 2-dimensional Hamiltonian flows the above identity yields*

$$\frac{1}{|\Lambda|} = \frac{1}{|\Lambda|(1 - 1/\Lambda)^2}(1 - 2/\Lambda + 1/\Lambda^2)\,,$$

*so*

$$1/\zeta = \frac{\det(1 - z\mathcal{L})\det(1 - z\mathcal{L}_{(2)})}{\det(1 - z\mathcal{L}_{(1)})}\,. \tag{19.26}$$

*This establishes that for nice* $2-dimensional$ *hyperbolic flows the dynamical zeta function is meromorphic.*

**Example 19.7** ***Dynamical zeta functions for*** $2-dimensional$ **Hamiltonian flows:** *The relation (19.26) is not particularly useful for our purposes. Instead we insert the identity*

$$1 = \frac{1}{(1 - 1/\Lambda)^2} - \frac{2}{\Lambda}\frac{1}{(1 - 1/\Lambda)^2} + \frac{1}{\Lambda^2}\frac{1}{(1 - 1/\Lambda)^2}$$

*into the exponential representation (19.14) of* $1/\zeta_k$*, and obtain*

$$1/\zeta_k = \frac{\det(1 - z\mathcal{L}_{(k)})\det(1 - z\mathcal{L}_{(k+2)})}{\det(1 - z\mathcal{L}_{(k+1)})^2}\,. \tag{19.27}$$

*Even though we have no guarantee that* $\det(1 - z\mathcal{L}_{(k)})$ *are entire, we do know that the upper bound on the leading zeros of* $\det(1 - z\mathcal{L}_{(k+1)})$ *lies strictly below the leading zeros of* $\det(1 - z\mathcal{L}_{(k)})$*, and therefore we expect that for 2-dimensional Hamiltonian flows the dynamical zeta function* $1/\zeta_k$ *generically has a* double *leading pole coinciding with the leading zero of the* $\det(1 - z\mathcal{L}_{(k+1)})$ *spectral determinant. This might fail if the poles and leading eigenvalues come in wrong order, but we have not encountered such situations in our numerical investigations. This result can also be stated as follows: the theorem establishes that the spectral determinant (19.13) is entire, and also implies that the poles in* $1/\zeta_k$ *must have the right multiplicities to cancel in the* $\det(1 - z\mathcal{L}) = \prod 1/\zeta_k^{k+1}$ *product.*

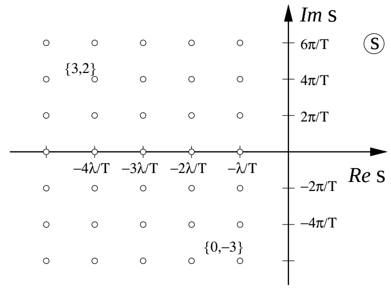



**Figure 19.3:** The classical resonances $\alpha = \{k, n\}$ (19.28) for a 2-disk game of pinball.

## 19.6 All too many eigenvalues?

What does the 2-dimensional hyperbolic Hamiltonian flow spectral determinant (19.13) tell us? Consider one of the simplest conceivable hyperbolic flows: the game of pinball of figure 19.2 consisting of two disks of equal size in a plane. There is only one periodic orbit, with the period $T$ and expanding eigenvalue $\Lambda$ given by elementary considerations (see exercise 13.7), and the resonances $\det(s_\alpha - \mathcal{A}) = 0$, $\alpha = \{k, n\}$ plotted in figure 19.3:

$$s_\alpha = -(k+1)\lambda + n\frac{2\pi i}{T}\,, \quad n \in \mathbb{Z}\,, \; k \in \mathbb{Z}_+\,, \quad \text{multiplicity } k+1\,, \qquad (19.28)$$

can be read off the spectral determinant (19.13) for a single unstable cycle:

$$\det(s - \mathcal{A}) = \prod_{k=0}^{\infty} \left(1 - e^{-sT}/|\Lambda|\Lambda^k\right)^{k+1} \,. \qquad (19.29)$$

In the above $\lambda = \ln|\Lambda|/T$ is the cycle Lyapunov exponent. For an open system, the real part of the eigenvalue $s_\alpha$ gives the decay rate of $\alpha$th eigenstate, and the imaginary part gives the "node number" of the eigenstate. The negative real part of $s_\alpha$ indicates that the resonance is unstable, and the decay rate in this simple case (zero entropy) equals the cycle Lyapunov exponent.

Rapidly decaying eigenstates with large negative $\text{Re}\, s_\alpha$ are not a problem, but as there are eigenvalues arbitrarily far in the imaginary direction, this might seem like all too many eigenvalues. However, they are necessary - we can check this by explicit computation of the right hand side of (18.23), the trace formula for flows:

$$\begin{aligned}
\sum_{\alpha=0}^{\infty} e^{s_\alpha t} &= \sum_{k=0}^{\infty} \sum_{n=-\infty}^{\infty} (k+1) e^{-(k+1)\lambda t + i2\pi n t/T} \\
&= \sum_{k=0}^{\infty} (k+1) \left(\frac{1}{|\Lambda|\Lambda^k}\right)^{t/T} \sum_{n=-\infty}^{\infty} e^{i2\pi n t/T} \\
&= \sum_{k=0}^{\infty} \frac{k+1}{|\Lambda|^r \Lambda^{kr}} \sum_{r=-\infty}^{\infty} \delta(r - t/T) \\
&= T \sum_{r=-\infty}^{\infty} \frac{\delta(t - rT)}{|\Lambda|^r (1 - 1/\Lambda^r)^2} \,. \qquad (19.30)
\end{aligned}$$

Hence, the two sides of the trace formula (18.23) are verified. The formula is fine for $t > 0$; for $t \to 0_+$, however, sides are divergent and need regularization.

The reason why such sums do not occur for maps is that for discrete time we work with the variable $z = e^s$, so an infinite strip along Im $s$ maps into an annulus in the complex $z$ plane, and the Dirac delta sum in the above is replaced by the Kronecker delta sum in (18.8). In the case at hand there is only one time scale $T$, and we could just as well replace $s$ by the variable $z = e^{-sT}$. In general, a continuous time flow has an infinity of irrationally related cycle periods, and the resonance arrays are more irregular, *cf.* figure 20.1.

## Résumé


The eigenvalues of evolution operators are given by the zeros of corresponding determinants, and one way to evaluate determinants is to expand them in terms of traces, using the matrix identity log det = tr log. Traces of evolution operators can be evaluated as integrals over Dirac delta functions, and in this way the spectra of evolution operators are related to periodic orbits. The spectral problem is now recast into a problem of determining zeros of either the *spectral determinant*

$$\det(s - \mathcal{A}) = \exp\left( -\sum_p \sum_{r=1}^{\infty} \frac{1}{r} \frac{e^{(\beta \cdot A_p - sT_p)r}}{\left|\det\left(\mathbf{1} - M_p^r\right)\right|} \right),$$

or the leading zeros of the *dynamical zeta function*

$$1/\zeta = \prod_p \left(1 - t_p\right), \qquad t_p = \frac{1}{|\Lambda_p|} e^{\beta \cdot A_p - sT_p}.$$

The spectral determinant is the tool of choice in actual calculations, as it has superior convergence properties (this will be discussed in chapter 23 and is illustrated, for example, by table 20.2). In practice both spectral determinants and dynamical zeta functions are preferable to trace formulas because they yield the eigenvalues more readily; the main difference is that while a trace diverges at an eigenvalue and requires extrapolation methods, determinants vanish at $s$ corresponding to an eigenvalue $s_\alpha$, and are analytic in $s$ in an open neighborhood of $s_\alpha$.

The critical step in the derivation of the periodic orbit formulas for spectral determinants and dynamical zeta functions is the hyperbolicity assumption (18.5) that no cycle stability eigenvalue is marginal, $|\Lambda_{p,i}| \neq 1$. By dropping the prefactors in (1.5), we have given up on any possibility of recovering the precise distribution of the initial $x$ (return to the past is rendered moot by the chaotic mixing and the exponential growth of errors), but in exchange we gain an effective description of the asymptotic behavior of the system. The pleasant surprise (to be demonstrated in chapter 20) is that the infinite time behavior of an unstable system turns out to be as easy to determine as its short time behavior.

## Commentary

**Remark 19.1** Piecewise monotone maps. A partial list of cases for which the transfer operator is well defined: the expanding Hölder case, weighted subshifts of finite type, expanding differentiable case, see Bowen [1.28]: expanding holomorphic case, see Ruelle [23.9]; piecewise monotone maps of the interval, see Hofbauer and Keller [19.13] and Baladi and Keller [19.16].

**Remark 19.2** Smale's wild idea. Smale's wild idea quoted on page 364 was technically wrong because 1) the Selberg zeta function yields the spectrum of a quantum mechanical Laplacian rather than the classical resonances, 2) the spectral determinant weights are different from what Smale conjectured, as the individual cycle weights also depend on the stability of the cycle, 3) the formula is not dimensionally correct, as $k$ is an integer and $s$ represents inverse time. Only for spaces of constant negative curvature do all cycles have the same Lyapunov exponent $\lambda = \ln|\Lambda_p|/T_p$. In this case, one can normalize time so that $\lambda = 1$, and the factors $e^{-sT_p}/\Lambda_p^k$ in (19.9) simplify to $s^{-(s+k)T_p}$, as intuited in Smale's quote on page 364 (where $l(\gamma)$ is the cycle period denoted here by $T_p$). Nevertheless, Smale's intuition was remarkably on the target.

**Remark 19.3** Is this a generalization of the Fourier analysis? Fourier analysis is a theory of the space $\leftrightarrow$ eigenfunction duality for dynamics on a circle. The way in which periodic orbit theory generalizes Fourier analysis to nonlinear flows is discussed in ref. [19.3], a very readable introduction to the Selberg Zeta function.

**Remark 19.4** Zeta functions, antecedents. For a function to be deserving of the appellation "zeta function," one expects it to have an Euler product representation (19.15), and perhaps also satisfy a functional equation. Various kinds of zeta functions are reviewed in refs. [19.6, 19.7, 19.8]. Historical antecedents of the dynamical zeta function are the fixed-point counting functions introduced by Weil [19.9], Lefschetz [19.10] and Artin and Mazur [19.11], and the determinants of transfer operators of statistical mechanics [1.29].

In his review article Smale [1.27] already intuited, by analogy to the Selberg Zeta function, that the spectral determinant is the right generalization for continuous time flows. In dynamical systems theory, dynamical zeta functions arise naturally only for piecewise linear mappings; for smooth flows the natural object for the study of classical and quantal spectra are the spectral determinants. Ruelle derived the relation (19.3) between spectral determinants and dynamical zeta functions, but since he was motivated by the Artin-Mazur zeta function (15.27) and the statistical mechanics analogy, he did not consider the spectral determinant to be a more natural object than the dynamical zeta function. This has been put right in papers on "flat traces" [11.18, 23.23].

The nomenclature has not settled down yet; what we call evolution operators here is elsewhere called transfer operators [1.32], Perron-Frobenius operators [19.4] and/or Ruelle-Araki operators.

Here we refer to kernels such as (17.23) as evolution operators. We follow Ruelle in usage of the term "dynamical zeta function," but elsewhere in the literature the function

(19.15) is often called the Ruelle zeta function. Ruelle [1.33] points out that the corresponding transfer operator $T$ was never considered by either Perron or Frobenius; a more appropriate designation would be the Ruelle-Araki operator. Determinants similar to or identical with our spectral determinants are sometimes called Selberg Zetas, Selberg-Smale zetas [1.8], functional determinants, Fredholm determinants, or even - to maximize confusion - dynamical zeta functions [19.12]. A Fredholm determinant is a notion that applies only to trace class operators - as we consider here a somewhat wider class of operators, we prefer to refer to their determinants loosely as “spectral determinants.”

## Exercises

19.1. **Escape rate for a** $1-$ *dimensional* **repeller, numerically.** Consider the quadratic map

$$f(x) = Ax(1-x) \tag{19.31}$$

on the unit interval. The trajectory of a point starting in the unit interval either stays in the interval forever or after some iterate leaves the interval and diverges to minus infinity. Estimate numerically the escape rate (22.8), the rate of exponential decay of the measure of points remaining in the unit interval, for either $A = 9/2$ or $A = 6$. Remember to compare your numerical estimate with the solution of the continuation of this exercise, exercise 20.2.

19.2. **Spectrum of the “golden mean” pruned map.** (medium - exercise 15.7 continued)

(a) Determine an expression for $\mathrm{tr}\,\mathcal{L}^n$, the trace of powers of the Perron-Frobenius operator (16.10) acting on the space of real analytic functions for the tent map of exercise 15.7.

(b) Show that the spectral determinant for the Perron-Frobenius operator is

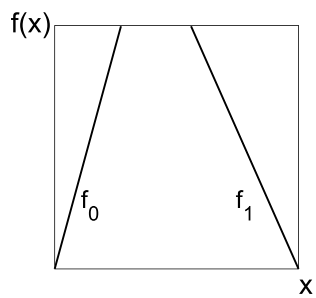
$$\det(1-z\mathcal{L}) = \prod_{k\ \mathrm{even}}\left(1-\frac{z}{\Lambda^{k+1}}-\frac{z^2}{\Lambda^{2k+2}}\right) \times \prod_{k\ \mathrm{odd}}\left(1+\frac{z}{\Lambda^{k+1}}+\frac{z^2}{\Lambda^{2k+2}}\right). \tag{19.32}$$

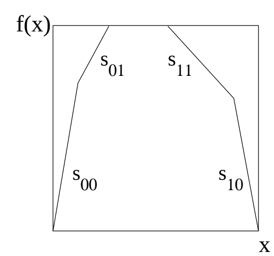
19.3. **Dynamical zeta functions.** (easy)

(a) Evaluate in closed form the dynamical zeta function

$$1/\zeta(z) = \prod_p \left(1-\frac{z^{n_p}}{|\Lambda_p|}\right)$$

for the piecewise-linear map (17.17) with the left branch slope $\Lambda_0$, the right branch slope $\Lambda_1$.




(b) What if there are four different slopes $s_{00}, s_{01}, s_{10}$, and $s_{11}$ instead of just two, with the preimages of the gap adjusted so that junctions of branches $s_{00}, s_{01}$ and $s_{11}, s_{10}$ map in the gap in one iteration? What would the dynamical zeta function be?

19.4. **Dynamical zeta functions from transition graphs.** Extend sect. 15.3 to evaluation of dynamical zeta functions for piecewise linear maps with finite transition graphs. This generalizes the results of exercise 19.3.

19.5. **Zeros of infinite products.** Determination of the quantities of interest by periodic orbits involves working with infinite product formulas.

(a) Consider the infinite product

$$F(z) = \prod_{k=0}^{\infty}(1+f_k(z))$$

where the functions $f_k$ are "sufficiently nice." This infinite product can be converted into an infinite sum by the use of a logarithm. Use the properties of infinite sums to develop a sensible definition of infinite products.

(b) If $z^*$ is a root of the function $F$, show that the infinite product diverges when evaluated at $z^*$.

(c) How does one compute a root of a function represented as an infinite product?

(d) Let $p$ be all prime cycles of the binary alphabet $\{0,1\}$. Apply your definition of $F(z)$ to the infinite product

$$F(z) = \prod_p (1 - \frac{z^{n_p}}{\Lambda^{n_p}})$$

(e) Are the roots of the factors in the above product the zeros of $F(z)$?

(Per Rosenqvist)

19.6. **Dynamical zeta functions as ratios of spectral determinants.** (medium) Show that the zeta function

$$1/\zeta(z) = \exp\left( - \sum_p \sum_{r=1} \frac{1}{r} \frac{z^{n_p}}{|\Lambda_p|^r} \right)$$

can be written as the ratio

$$1/\zeta(z) = \det(1 - z\mathcal{L}_{(0)})/\det(1 - z\mathcal{L}_{(1)}) ,$$

where $\det(1 - z\mathcal{L}_{(s)}) = \prod_p \prod_{k=0}^{\infty}(1 - z^{n_p}/|\Lambda_p|\Lambda_p^{k+s})$.

19.7. **Contour integral for survival probability.** Perform explicitly the contour integral appearing in (19.19).

19.8. **Dynamical zeta function for maps.** In this problem we will compare the dynamical zeta function and the spectral determinant. Compute the exact dynamical zeta function for the skew full tent map (16.45)

$$1/\zeta(z) = \prod_{p \in P} \left( 1 - \frac{z^{n_p}}{|\Lambda_p|} \right) .$$

What are its roots? Do they agree with those computed in exercise 16.7?

19.9. **Dynamical zeta functions for Hamiltonian maps.** Starting from

$$1/\zeta(s) = \exp\left( - \sum_p \sum_{r=1}^{\infty} \frac{1}{r} t_p^r \right)$$

for a 2-dimensional Hamiltonian map. Using the equality

$$1 = \frac{1}{(1 - 1/\Lambda)^2}(1 - 2/\Lambda + 1/\Lambda^2) ,$$

show that

$$1/\zeta = \det(1 - \mathcal{L})\det(1 - \mathcal{L}_{(2)})/\det(1 - \mathcal{L}_{(1)})^2 .$$

In this expression $\det(1 - z\mathcal{L}_{(k)})$ is the expansion one gets by replacing $t_p \to t_p/\Lambda_p^k$ in the spectral determinant.

19.10. **Riemann $\zeta$ function.** The Riemann $\zeta$ function is defined as the sum

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} , \qquad s \in \mathbb{C} .$$

(a) Use factorization into primes to derive the Euler product representation

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}} .$$

The dynamical zeta function exercise 19.15 is called a "zeta" function because it shares the form of the Euler product representation with the Riemann zeta function.

(b) (Not trivial:) For which complex values of $s$ is the Riemann zeta sum convergent?

(c) Are the zeros of the terms in the product, $s = -\ln p$, also the zeros of the Riemann $\zeta$ function? If not, why not?

19.11. **Finite truncations.** (easy) Suppose we have a 1-dimensional system with complete binary dynamics, where the stability of each orbit is given by a simple multiplicative rule:

$$\Lambda_p = \Lambda_0^{n_{p,0}} \Lambda_1^{n_{p,1}} , \qquad n_{p,0} = \#0 \text{ in } p , \; n_{p,1} = \#1 \text{ in } p ,$$

so that, for example, $\Lambda_{00101} = \Lambda_0^3 \Lambda_1^2$.

(a) Compute the dynamical zeta function for this system; perhaps by creating a transfer matrix analogous to (17.19), with the right weights.

(b) Compute the finite $p$ truncations of the cycle expansion, i.e. take the product only over the $p$ up to given length with $n_p \le N$, and expand as a series in $z$

$$\prod_p \left( 1 - \frac{z^{n_p}}{|\Lambda_p|} \right) .$$

Do they agree? If not, how does the disagreement depend on the truncation length $N$?