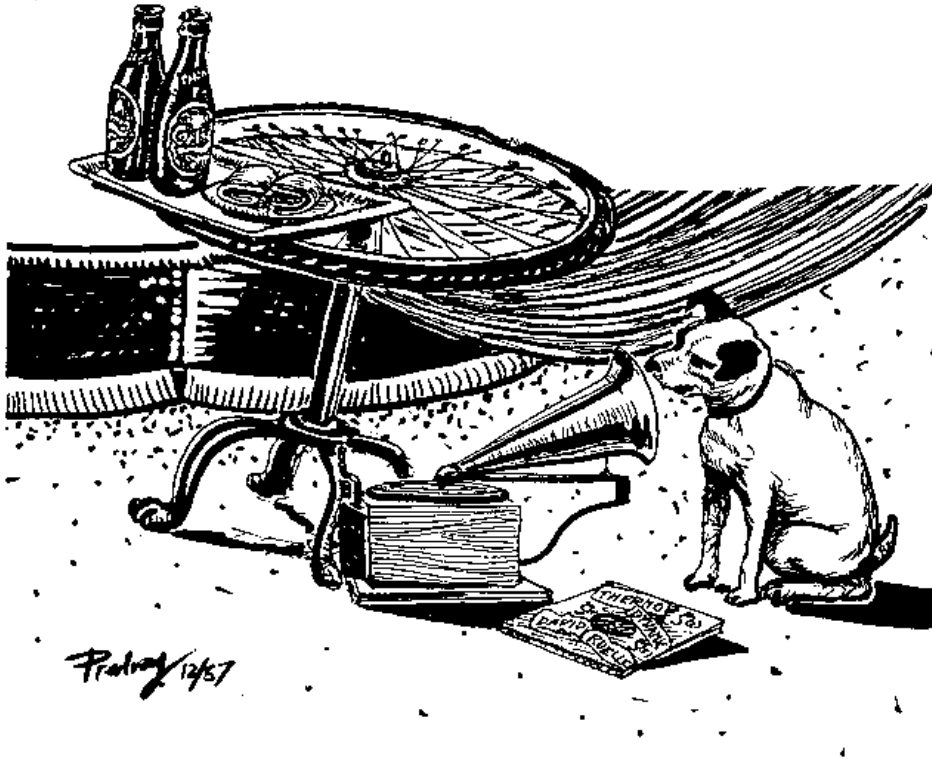


# Chaos: Classical and Quantum

## I: Deterministic Chaos



Predrag Cvitanović – Roberto Artuso – Ronnie Mainieri – Gregor Tanner –  
Gábor Vattay

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# Contributors

No man but a blockhead ever wrote except for money

—Samuel Johnson

This book is a result of collaborative labors of many people over a span of several decades. Coauthors of a chapter or a section are indicated in the byline to the chapter/section title. If you are referring to a specific coauthored section rather than the entire book, cite it as (for example):

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### Gregor Tanner

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Daniel Borrero Oct 23 2008, `soluCycles.tex`

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I feel I never want to write another book. What's the good!  
I can eke living on stories and little articles, that don't cost  
a tithe of the output a book costs. Why write novels any  
more!

—D.H. Lawrence

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F. Haake's heartfelt lament on page 355 was uttered at the end of the first conference presentation of cycle expansions, in 1988. G.P. Morriss advice to students as how to read the introduction to this book, page 6, was offered during a 2002 graduate course in Dresden. K. Huang's C.N. Yang interview quoted on page 317 is available on [ChaosBook.org/extras](http://ChaosBook.org/extras). T.D. Lee remarks on as to who is to blame, page 37 and page 251, as well as M. Shub's helpful technical remark on page 453 came during the Rockefeller University December 2004 "Feigenbaum Fest." Quotes on pages 37, 121, and 314 are taken from a book review by J. Guckenheimer [1.1].

Who is the 3-legged dog reappearing throughout the book? Long ago, when we were innocent and knew not Borel measurable  $\alpha$  to  $\Omega$  sets, P. Cvitanović asked V. Baladi a question about dynamical zeta functions, who then asked J.-P. Eckmann, who then asked D. Ruelle. The answer was transmitted back: "The master

says: ‘It is holomorphic in a strip’.” Hence His Master’s Voice logo, and the 3-legged dog is us, still eager to fetch the bone. The answer has made it to the book, though not precisely in His Master’s voice. As a matter of fact, the answer *is* the book. We are still chewing on it.

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