

Chaos: Classical and Quantum

I: Deterministic Chaos



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Contents

Contributors	xi
Acknowledgements	xv
I Geometry of chaos	1
1 Overture	3
1.1 Why ChaosBook?	4
1.2 Chaos ahead	5
1.3 The future as in a mirror	6
1.4 A game of pinball	11
1.5 Chaos for cyclists	15
1.6 Change in time	21
1.7 To statistical mechanics	24
1.8 Chaos: what is it good for?	25
1.9 What is not in ChaosBook	27
résumé 28 commentary 30 guide to exercises 33 exercises 34 references 34	
2 Go with the flow	37
2.1 Dynamical systems	37
2.2 Flows	42
2.3 Computing trajectories	46
résumé 47 commentary 47 exercises 50 references 51	
3 Discrete time dynamics	54
3.1 Poincaré sections	54
3.2 Constructing a Poincaré section	61
3.3 Maps	62
résumé 65 commentary 66 exercises 67 references 67	
4 Local stability	69
4.1 Flows transport neighborhoods	69
4.2 Linear flows	73
4.3 Stability of flows	79
4.4 Neighborhood volume	84
4.5 Stability of maps	86
résumé 89 commentary 89 exercises 90 references 91	

5	Cycle stability	94
5.1	Stability of periodic orbits	94
5.2	Floquet multipliers are invariant	99
5.3	Stability of Poincaré map cycles	101
5.4	There goes the neighborhood	102
	résumé 102 commentary 103 exercises 104 references 104	
6	Get straight	105
6.1	Changing coordinates	105
6.2	Rectification of flows	107
6.3	Collinear helium	108
6.4	Rectification of maps	113
6.5	Rectification of a periodic orbit	114
6.6	Cycle Floquet multipliers are metric invariants	115
	résumé 116 commentary 117 exercises 118 references 118	
7	Hamiltonian dynamics	120
7.1	Hamiltonian flows	121
7.2	Stability of Hamiltonian flows	122
7.3	Symplectic maps	125
7.4	Poincaré invariants	128
	commentary 129 exercises 131 references 131	
8	Billiards	133
8.1	Billiard dynamics	133
8.2	Stability of billiards	135
	résumé 138 commentary 138 exercises 139 references 139	
9	World in a mirror	142
9.1	Discrete symmetries	142
9.2	Symmetries of solutions	150
9.3	Relative periodic orbits	154
9.4	Dynamics reduced to fundamental domain	156
	résumé 159 commentary 160 exercises 162 references 163	
10	Relativity for cyclists	166
10.1	Continuous symmetries	166
10.2	Symmetries of solutions	174
10.3	Stability	180
10.4	Reduced state space	180
10.5	Method of images: Hilbert bases	186
	résumé 189 commentary 191 exercises 194 references 197	
11	Charting the state space	202
11.1	Qualitative dynamics	203
11.2	Stretch and fold	207
11.3	Temporal ordering: itineraries	210
11.4	Spatial ordering	212
11.5	Kneading theory	216

11.6 Symbolic dynamics, basic notions	218
résumé 221 commentary 222 exercises 224 references 225	
12 Stretch, fold, prune	227
12.1 Going global: stable/unstable manifolds	228
12.2 Horseshoes	232
12.3 Symbol plane	236
12.4 Prune danish	239
12.5 Recoding, symmetries, tilings	240
résumé 243 commentary 244 exercises 246 references 247	
13 Fixed points, and how to get them	251
13.1 Where are the cycles?	252
13.2 One-dimensional mappings	256
13.3 Multipoint shooting method	258
13.4 Flows	260
résumé 264 commentary 265 exercises 266 references 268	
II Chaos rules	270
14 Walkabout: Transition graphs	272
14.1 Matrix representations of topological dynamics	272
14.2 Transition graphs: wander from node to node	274
14.3 Transition graphs: stroll from link to link	277
résumé 281 commentary 281 exercises 283 references 283	
15 Counting	285
15.1 How many ways to get there from here?	286
15.2 Topological trace formula	288
15.3 Determinant of a graph	291
15.4 Topological zeta function	295
15.5 Infinite partitions	297
15.6 Shadowing	299
15.7 Counting cycles	300
résumé 304 commentary 305 exercises 306 references 309	
16 Transporting densities	310
16.1 Measures	311
16.2 Perron-Frobenius operator	312
16.3 Why not just leave it to a computer?	314
16.4 Invariant measures	317
16.5 Density evolution for infinitesimal times	320
16.6 Liouville operator	322
résumé 324 commentary 325 exercises 326 references 328	

17 Averaging	330
17.1 Dynamical averaging	330
17.2 Evolution operators	338
17.3 Lyapunov exponents	340
résumé 344 commentary 344 exercises 345 references 346	
18 Trace formulas	349
18.1 A trace formula for maps	350
18.2 A trace formula for flows	355
18.3 An asymptotic trace formula	358
résumé 359 commentary 360 exercises 360 references 361	
19 Spectral determinants	362
19.1 Spectral determinants for maps	362
19.2 Spectral determinant for flows	364
19.3 Dynamical zeta functions	366
19.4 False zeros	370
19.5 Spectral determinants vs. dynamical zeta functions	370
19.6 All too many eigenvalues?	372
résumé 373 commentary 374 exercises 375 references 376	
20 Cycle expansions	378
20.1 Pseudocycles and shadowing	378
20.2 Construction of cycle expansions	381
20.3 Cycle formulas for dynamical averages	385
20.4 Cycle expansions for finite alphabets	388
20.5 Stability ordering of cycle expansions	389
20.6 Dirichlet series	392
résumé 393 commentary 394 exercises 396 references 398	
21 Discrete factorization	400
21.1 Preview	401
21.2 Discrete symmetries	403
21.3 Dynamics in the fundamental domain	404
21.4 Factorizations of dynamical zeta functions	408
21.5 C_2 factorization	410
21.6 D_3 factorization: 3-disk game of pinball	411
résumé 413 commentary 414 exercises 414 references 415	
III Chaos: what to do about it?	417
22 Why cycle?	419
22.1 Escape rates	419
22.2 Natural measure in terms of periodic orbits	421
22.3 Flow conservation sum rules	423
22.4 Correlation functions	424
22.5 Trace formulas vs. level sums	425
résumé 427 commentary 427 exercises 428 references 429	

23 Why does it work?	431
23.1 Linear maps: exact spectra	432
23.2 Evolution operator in a matrix representation	436
23.3 Classical Fredholm theory	439
23.4 Analyticity of spectral determinants	441
23.5 Hyperbolic maps	446
23.6 The physics of eigenvalues and eigenfunctions	448
23.7 Troubles ahead	450
résumé 451 commentary 453 exercises 455 references 455	
24 Intermittency	458
24.1 Intermittency everywhere	459
24.2 Intermittency for pedestrians	462
24.3 Intermittency for cyclists	473
24.4 BER zeta functions	479
résumé 481 commentary 482 exercises 483 references 484	
25 Deterministic diffusion	486
25.1 Diffusion in periodic arrays	487
25.2 Diffusion induced by chains of 1 – <i>dimensional</i> maps	491
25.3 Marginal stability and anomalous diffusion	498
résumé 501 commentary 502 exercises 504 references 504	
26 Turbulence?	507
26.1 Fluttering flame front	508
26.2 Infinite-dimensional flows: Numerics	511
26.3 Visualization	513
26.4 Equilibria of equilibria	514
26.5 Why does a flame front flutter?	516
26.6 Periodic orbits	519
26.7 Intrinsic parametrization	519
26.8 Energy budget	520
résumé 522 commentary 523 exercises 524 references 524	
27 Irrationally winding	526
27.1 Mode locking	527
27.2 Local theory: “Golden mean” renormalization	532
27.3 Global theory: Thermodynamic averaging	534
27.4 Hausdorff dimension of irrational windings	536
27.5 Thermodynamics of Farey tree: Farey model	538
résumé 540 commentary 540 exercises 543 references 544	
IV The rest is noise	547
28 Noise	549
28.1 Deterministic transport	550
28.2 Brownian diffusion	551
28.3 Weak noise	552

28.4	Weak noise approximation	554
	résumé 555 commentary 555 exercises 557 references 557	
29	Relaxation for cyclists	560
29.1	Fictitious time relaxation	561
29.2	Discrete iteration relaxation method	566
29.3	Least action method	570
	résumé 570 commentary 571 exercises 574 references 574	
V	Quantum chaos	577
30	Prologue	579
30.1	Quantum pinball	580
30.2	Quantization of helium	582
	commentary 583 references 584	
31	Quantum mechanics, briefly	585
	exercises 589	
32	WKB quantization	590
32.1	WKB ansatz	590
32.2	Method of stationary phase	593
32.3	WKB quantization	594
32.4	Beyond the quadratic saddle point	596
	résumé 597 commentary 598 exercises 598 references 599	
33	Semiclassical evolution	600
33.1	Hamilton-Jacobi theory	600
33.2	Semiclassical propagator	609
33.3	Semiclassical Green's function	612
	résumé 618 commentary 619 exercises 621 references 622	
34	Semiclassical quantization	623
34.1	Trace formula	623
34.2	Semiclassical spectral determinant	629
34.3	One-dof systems	630
34.4	Two-dof systems	631
	résumé 632 commentary 633 exercises 635 references 635	
35	Quantum scattering	637
35.1	Density of states	637
35.2	Quantum mechanical scattering matrix	641
35.3	Krein-Friedel-Lloyd formula	642
35.4	Wigner time delay	645
	commentary 647 exercises 648 references 648	

36 Chaotic multiscattering	651
36.1 Quantum mechanical scattering matrix	652
36.2 N -scatterer spectral determinant	655
36.3 Semiclassical 1-disk scattering	659
36.4 From quantum cycle to semiclassical cycle	666
36.5 Heisenberg uncertainty	669
commentary 669 references 670	
37 Helium atom	671
37.1 Classical dynamics of collinear helium	672
37.2 Chaos, symbolic dynamics and periodic orbits	673
37.3 Local coordinates, Jacobian matrix	677
37.4 Getting ready	679
37.5 Semiclassical quantization of collinear helium	681
résumé 688 commentary 688 exercises 690 references 691	
38 Diffraction distraction	692
38.1 Quantum eavesdropping	692
38.2 An application	698
résumé 703 commentary 704 exercises 705 references 706	
Epilogue	708
Index	713

A	A brief history of chaos	732
A.1	Chaos is born	732
A.2	Chaos with us	737
A.3	Death of the Old Quantum Theory	744
	commentary 747 references 749	
B	Linear stability	752
B.1	Linear algebra	752
B.2	Eigenvalues and eigenvectors	754
B.3	Eigenspectra: what to make out of them?	761
B.4	Stability of Hamiltonian flows	762
B.5	Monodromy matrix for Hamiltonian flows	764
	exercises 767 references 767	
C	Finding cycles	768
C.1	Newton-Raphson method	768
C.2	Hybrid Newton-Raphson / relaxation method	769
D	Symbolic dynamics techniques	772
D.1	Topological zeta functions for infinite subshifts	772
D.2	Prime factorization for dynamical itineraries	780
E	Counting itineraries	784
E.1	Counting curvatures	784
	exercises 785	
F	Implementing evolution	786
F.1	Koopmania	786
F.2	Implementing evolution	788
	commentary 791 exercises 791 references 792	
G	Transport of vector fields	794
G.1	Evolution operator for Lyapunov exponents	794
G.2	Advection of vector fields by chaotic flows	799
	commentary 803 exercises 803 references 803	
H	Discrete symmetries of dynamics	805
H.1	Preliminaries and definitions	805
H.2	Invariants and reducibility	812
H.3	Lattice derivatives	815
H.4	Periodic lattices	818
H.5	Discrete Fourier transforms	820
H.6	C_{4v} factorization	823
H.7	C_{2v} factorization	827
H.8	Hénon map symmetries	830
	commentary 830 exercises 831 references 833	

I	Convergence of spectral determinants	835
I.1	Curvature expansions: geometric picture	835
I.2	On importance of pruning	838
I.3	Ma-the-matical caveats	839
I.4	Estimate of the n th cumulant	840
J	Infinite dimensional operators	842
J.1	Matrix-valued functions	842
J.2	Operator norms	844
J.3	Trace class and Hilbert-Schmidt class	845
J.4	Determinants of trace class operators	847
J.5	Von Koch matrices	850
J.6	Regularization	852
	exercises 854 references 854	
K	Thermodynamic formalism	856
K.1	Rényi entropies	856
K.2	Fractal dimensions	861
	résumé 865 commentary 865 exercises 866 references 866	
L	Statistical mechanics recycled	868
L.1	The thermodynamic limit	868
L.2	Ising models	870
L.3	Fisher droplet model	874
L.4	Scaling functions	879
L.5	Geometrization	882
	résumé 889 commentary 890 exercises 890 references 891	
M	Noise/quantum corrections	893
M.1	Periodic orbits as integrable systems	893
M.2	The Birkhoff normal form	897
M.3	Bohr-Sommerfeld quantization of periodic orbits	898
M.4	Quantum calculation of \hbar corrections	900
	references 906	
S	Solutions	909
T	Projects	1002
T.1	Deterministic diffusion, zig-zag map	1004
	references 1009	
T.2	Deterministic diffusion, sawtooth map	1010

Contributors

No man but a blockhead ever wrote except for money

—Samuel Johnson

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Roberto Artuso

16 Transporting densities	310
18.2 A trace formula for flows	355
22.4 Correlation functions	424
24 Intermittency	458
25 Deterministic diffusion	486

Ronnie Mainieri

2 Flows	37
3.2 The Poincaré section of a flow	61
4 Local stability	69
6.1 Understanding flows	108
11.1 Temporal ordering: itineraries	203
Appendix A: A brief history of chaos	732

Gábor Vattay

Gregor Tanner

24 Intermittency	458
Appendix B.5: Jacobians of Hamiltonian flows	764

Arindam Basu

Rössler flow figures, tables, cycles in chapters 11, 13 and exercise 13.10

Ofer Biham

29.1 Cyclists relaxation method	561
---------------------------------------	-----

Daniel Borrero Oct 23 2008, `soluCycles.tex`

Solution 13.15

Cristel Chandre	
29.1 Cyclists relaxation method	561
29.2 Discrete cyclists relaxation methods	566
Freddy Christiansen	
13.2 One-dimensional mappings	256
13.3 Multipoint shooting method	258
Per Dahlqvist	
24 Intermittency	458
29.3 Orbit length extremization method for billiards	570
Carl P. Dettmann	
20.5 Stability ordering of cycle expansions	389
Fotis K. Diakonou	
29.2 Discrete cyclists relaxation methods	566
G. Bard Ermentrout	
Exercise 5.1	
Mitchell J. Feigenbaum	
Appendix B.4: Symplectic invariance	762
Jonathan Halcrow	
Example 3.5: Sections of Lorenz flow	60
Example 4.7: Stability of Lorenz flow equilibria	81
Example 4.8: Lorenz flow: Global portrait	83
Example 9.10: Desymmetrization of Lorenz flow	152
Example 11.4: Lorenz flow: a 1-dimensional return map	208
Exercises 9.9 and figure 2.5	
Kai T. Hansen	
11.3 Unimodal map symbolic dynamics	210
15.5 Topological zeta function for an infinite partition	297
11.5 Kneading theory	216
figures throughout the text	
Rainer Klages	
Figure 25.5	
Yueheng Lan	
Solutions 1.1, 2.1, 2.2, 2.3, 2.4, 2.5, 9.6, 12.6, 11.6, 16.1, 16.2, 16.3, 16.5, 16.7, 16.10, 17.1 and figures 1.9, 9.2, 9.8 11.5,	
Bo Li	
Solutions 31.2, 31.1, 32.1	
Joachim Mathiesen	
17.3 Lyapunov exponents	340

Rössler flow figures, tables, cycles in sect. 17.3 and exercise 13.10

Yamato Matsuoka

Figure 12.4

Radford Mitchell, Jr.

Example 3.6

Rytis Paškauskas

4.5.1 Stability of Poincaré return maps 87

5.3 Stability of Poincaré map cycles 101

Exercises 2.8, 3.1, 4.4 and solution 4.1

Adam Prügel-Bennet

Solutions 1.2, 2.10, 8.1, 19.1, 20.2 23.3, 29.1.

Lamberto Rondoni

16 Transporting densities 310

13.1.1 Cycles from long time series 253

22.2.1 Unstable periodic orbits are dense 422

Table 15.2

Juri Rolf

Solution 23.3

Per E. Rosenqvist

exercises, figures throughout the text

Hans Henrik Rugh

23 Why does it work? 431

Peter Schmelcher

29.2 Discrete cyclists relaxation methods 566

Evangelos Siminos

Example 3.5: Sections of Lorenz flow 60

Example 4.7: Stability of Lorenz flow equilibria 81

Example 4.8: Lorenz flow: Global portrait 83

Example 9.10: Desymmetrization of Lorenz flow 152

Example 11.4: Lorenz flow: a 1-dimensional return map 208

Exercise 9.9

Solution 10.21

Gábor Simon

Rössler flow figures, tables, cycles in chapters 2, 13 and exercise 13.10

Edward A. Spiegel

2 Flows 37

16 Transporting densities 310

Luz V. Vela-Arevalo

7.1 Hamiltonian flows	121
Exercises 7.1, 7.3, 7.5	

R. Wilczak

Figure 10.1, figure 10.4

Exercise 10.26

Solutions 10.1, 10.5, 10.6, 10.7, 10.8, 10.10, 10.14, 10.15, 10.16, 10.17,
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I feel I never want to write another book. What's the good!
I can eke living on stories and little articles, that don't cost
a tithe of the output a book costs. Why write novels any
more!

—D.H. Lawrence

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Who is the 3-legged dog reappearing throughout the book? Long ago, when we were innocent and knew not Borel measurable α to Ω sets, P. Cvitanović asked V. Baladi a question about dynamical zeta functions, who then asked J.-P. Eckmann, who then asked D. Ruelle. The answer was transmitted back: "The

master says: 'It is holomorphic in a strip.'" Hence His Master's Voice logo, and the 3-legged dog is us, still eager to fetch the bone. The answer has made it to the book, though not precisely in His Master's voice. As a matter of fact, the answer *is* the book. We are still chewing on it.

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