

# Appendix H

## Discrete symmetries of dynamics

**B**ASIC GROUP-THEORETIC NOTIONS are recapitulated here: groups, irreducible representations, invariants. Our notation follows [birdtracks.eu](http://birdtracks.eu).

The key result is the construction of projection operators from invariant matrices. The basic idea is simple: a hermitian matrix can be diagonalized. If this matrix is an invariant matrix, it decomposes the reps of the group into direct sums of lower-dimensional reps. Most of computations to follow implement the spectral decomposition

$$\mathbf{M} = \lambda_1 \mathbf{P}_1 + \lambda_2 \mathbf{P}_2 + \cdots + \lambda_r \mathbf{P}_r,$$

which associates with each distinct root  $\lambda_i$  of invariant matrix  $\mathbf{M}$  a projection operator (H.17):

$$\mathbf{P}_i = \prod_{j \neq i} \frac{\mathbf{M} - \lambda_j \mathbf{1}}{\lambda_i - \lambda_j}.$$

Sects. H.3 and H.4 develop Fourier analysis as an application of the general theory of invariance groups and their representations.

### H.1 Preliminaries and definitions

(A. Wirzba and P. Cvitanović)

We define *group*, *representation*, *symmetry of a dynamical system*, and *invariance*.

**Group axioms.** A group  $G$  is a set of elements  $g_1, g_2, g_3, \dots$  for which *composition* or *group multiplication*  $g_2 \circ g_1$  (which we often abbreviate as  $g_2 g_1$ ) of any two elements satisfies the following conditions:

1. If  $g_1, g_2 \in G$ , then  $g_2 \circ g_1 \in G$ .
2. The group multiplication is associative:  $g_3 \circ (g_2 \circ g_1) = (g_3 \circ g_2) \circ g_1$ .
3. The group  $G$  contains *identity* element  $e$  such that  $g \circ e = e \circ g = g$  for every element  $g \in G$ .
4. For every element  $g \in G$ , there exists a unique  $h = g^{-1} \in G$  such that  $h \circ g = g \circ h = e$ .

A *finite* group is a group with a finite number of elements

$$G = \{e, g_2, \dots, g_{|G|}\},$$

where  $|G|$ , the number of elements, is the *order* of the group.

**Example H.1 Finite groups:** Some finite groups that frequently arise in applications:

- $C_n$  (also denoted  $Z_n$ ): the cyclic group of order  $n$ .
- $D_n$ : the dihedral group of order  $2n$ , rotations and reflections in plane that preserve a regular  $n$ -gon.
- $S_n$ : the symmetric group of all permutations of  $n$  symbols, order  $n!$ .

**Example H.2 Lie groups:** Some compact continuous groups that arise in dynamical systems applications:

- $S^1$  (also denoted  $T^1$ ): circle group of dimension 1.
- $T_m = S^1 \times S^1 \cdots \times S^1$ :  $m$ -torus, of dimension  $m$ .
- $SO(2)$ : rotations in the plane, dimension 1. Isomorphic to  $S^1$ .
- $O(2) = SO(2) \times D_1$ : group of rotations and reflections in the plane, of dimension 1.
- $U(1)$ : group of phase rotations in the complex plane, of dimension 1. Isomorphic to  $SO(2)$ .
- $SO(3)$ : rotation group of dimension 3.
- $SU(2)$ : unitary group of dimension 3. Isomorphic to  $SO(3)$ .
- $GL(n)$ : general linear group of invertible matrix transformations, dimension  $n^2$ .
- $SO(n)$ : special orthogonal group of dimension  $n(n-1)/2$ .
- $O(n) = SO(n) \times D_1$ : orthogonal group of dimension  $n(n-1)/2$ .
- $Sp(n)$ : symplectic group of dimension  $n(n+1)/2$ .
- $SU(n)$ : special unitary group of dimension  $n^2 - 1$ .

**Example H.3 Cyclic and dihedral groups:** The cyclic group  $C_n \subset SO(2)$  of order  $n$  is generated by one element. For example, this element can be rotation through  $2\pi/n$ . The dihedral group  $D_n \subset O(2)$ ,  $n > 2$ , can be generated by two elements one at least of which must reverse orientation. For example, take  $\sigma$  corresponding to reflection in the  $x$ -axis.  $\sigma^2 = e$ ; such operation  $\sigma$  is called an involution.  $C$  to rotation through  $2\pi/n$ , then  $D_n = \langle \sigma, C \rangle$ , and the defining relations are  $\sigma^2 = C^n = e$ ,  $(C\sigma)^2 = e$ .

Groups are defined and classified as abstract objects by their multiplication tables (for finite groups) or Lie algebras (for Lie groups). What concerns us in applications is their *action* as groups of transformations on a given space, usually a vector space (see appendix B.1), but sometimes an affine space, or a more general manifold  $\mathcal{M}$ .

**Repeated index summation.** Throughout this text, the repeated pairs of upper/lower indices are always summed over

$$G_a{}^b x_b \equiv \sum_{b=1}^n G_a{}^b x_b, \quad (\text{H.1})$$

unless explicitly stated otherwise.

**General linear transformations.** Let  $GL(n, \mathbb{F})$  be the group of general linear transformations,

$$GL(n, \mathbb{F}) = \{g : \mathbb{F}^n \rightarrow \mathbb{F}^n \mid \det(g) \neq 0\}. \quad (\text{H.2})$$

Under  $GL(n, \mathbb{F})$  a basis set of  $V$  is mapped into another basis set by multiplication with a  $[n \times n]$  matrix  $g$  with entries in field  $\mathbb{F}$  ( $\mathbb{F}$  is either  $\mathbb{R}$  or  $\mathbb{C}$ ),

$$\mathbf{e}'^a = \mathbf{e}^b (g^{-1})_b{}^a.$$

As the vector  $\mathbf{x}$  is what it is, regardless of a particular choice of basis, under this transformation its coordinates must transform as

$$x'_a = g_a{}^b x_b.$$

**Standard rep.** We shall refer to the set of  $[n \times n]$  matrices  $g$  as a *standard rep* of  $GL(n, \mathbb{F})$ , and the space of all  $n$ -tuples  $(x_1, x_2, \dots, x_n)^T$ ,  $x_i \in \mathbb{F}$  on which these matrices act as the *standard representation space*  $V$ .

Under a general linear transformation  $g \in GL(n, \mathbb{F})$ , the row of basis vectors transforms by right multiplication as  $\mathbf{e}' = \mathbf{e} g^{-1}$ , and the column of  $x_a$ 's transforms

by left multiplication as  $x' = gx$ . Under left multiplication the column (row transposed) of basis vectors  $\mathbf{e}^T$  transforms as  $\mathbf{e}'^T = g^\dagger \mathbf{e}^T$ , where the *dual rep*  $g^\dagger = (g^{-1})^T$  is the transpose of the inverse of  $g$ . This observation motivates introduction of a *dual* representation space  $\bar{V}$ , the space on which  $GL(n, \mathbb{F})$  acts via the dual rep  $g^\dagger$ .

**Dual space.** If  $V$  is a vector representation space, then the *dual space*  $\bar{V}$  is the set of all linear forms on  $V$  over the field  $\mathbb{F}$ .

If  $\{\mathbf{e}^{(1)}, \dots, \mathbf{e}^{(d)}\}$  is a (right) basis of  $V$ , then  $\bar{V}$  is spanned by the *dual basis* (left basis)  $\{\mathbf{e}_{(1)}, \dots, \mathbf{e}_{(d)}\}$ , the set of  $n$  linear forms  $\mathbf{e}_{(j)}$  such that

$$\mathbf{e}_{(i)} \cdot \mathbf{e}^{(j)} = \delta_i^j,$$

where  $\delta_a^b$  is the Kronecker symbol,  $\delta_a^b = 1$  if  $a = b$ , and zero otherwise. The components of dual representation space vectors will here be distinguished by upper indices

$$(y^1, y^2, \dots, y^n). \quad (\text{H.3})$$

They transform under  $GL(n, \mathbb{F})$  as

$$y'^a = (g^\dagger)_b^a y^b. \quad (\text{H.4})$$

For  $GL(n, \mathbb{F})$  no complex conjugation is implied by the  $\dagger$  notation; that interpretation applies only to unitary subgroups of  $GL(n, \mathbb{C})$ .  $g$  can be distinguished from  $g^\dagger$  by meticulously keeping track of the relative ordering of the indices,

$$g_a^b \rightarrow g_a^b, \quad (g^\dagger)_a^b \rightarrow g^b_a. \quad (\text{H.5})$$

**Defining space, dual space.** In what follows  $V$  will always denote the *defining*  $n$ -dimensional complex vector representation space, that is to say the initial, “elementary multiplet” space within which we commence our deliberations. Along with the defining vector representation space  $V$  comes the *dual*  $n$ -dimensional vector representation space  $\bar{V}$ . We shall denote the corresponding element of  $\bar{V}$  by raising the index, as in (H.3), so the components of defining space vectors, resp. dual vectors, are distinguished by lower, resp. upper indices:

$$\begin{aligned} x &= (x_1, x_2, \dots, x_n), & \mathbf{x} &\in V \\ \bar{x} &= (x^1, x^2, \dots, x^n), & \bar{\mathbf{x}} &\in \bar{V}. \end{aligned} \quad (\text{H.6})$$

**Defining rep.** Let  $G$  be a group of transformations acting linearly on  $V$ , with the action of a group element  $g \in G$  on a vector  $x \in V$  given by an  $[n \times n]$  matrix  $g$

$$x'_a = g_a^b x_b \quad a, b = 1, 2, \dots, n. \quad (\text{H.7})$$

We shall refer to  $g_a^b$  as the *defining rep* of the group  $G$ . The action of  $g \in G$  on a vector  $\bar{q} \in \bar{V}$  is given by the *dual rep*  $[n \times n]$  matrix  $g^\dagger$ :

$$x'^a = x^b (g^\dagger)_b^a = g^a_b x^b. \quad (\text{H.8})$$

In the applications considered here, the group  $G$  will almost always be assumed to be a subgroup of the *unitary group*, in which case  $g^{-1} = g^\dagger$ , and  $\dagger$  indicates hermitian conjugation:

$$(g^\dagger)_a^b = (g_b^a)^* = g^b_a. \quad (\text{H.9})$$

**Hermitian conjugation** is effected by complex conjugation and index transposition:

Complex conjugation interchanges upper and lower indices; transposition reverses their order. A matrix is *hermitian* if its elements satisfy

$$(\mathbf{M}^\dagger)_b^a = M_b^a. \quad (\text{H.10})$$

For a hermitian matrix there is no need to keep track of the relative ordering of indices, as  $M_b^a = (\mathbf{M}^\dagger)_b^a = M^a_b$ .

**Invariant vectors.** The vector  $q \in V$  is an *invariant vector* if for any transformation  $g \in G$

$$q = gq. \quad (\text{H.11})$$

If a bilinear form  $\mathbf{M}(\bar{x}, y) = x^a M_a^b y_b$  is invariant for all  $g \in G$ , the matrix

$$M_a^b = g_a^c g^b_d M_c^d \quad (\text{H.12})$$

is an *invariant matrix*. Multiplying with  $g_b^e$  and using the unitary condition (H.9), we find that the invariant matrices *commute* with all transformations  $g \in G$ :

$$[g, \mathbf{M}] = 0. \quad (\text{H.13})$$

**Invariants.** We shall refer to an invariant relation between  $p$  vectors in  $V$  and  $q$  vectors in  $\bar{V}$ , which can be written as a homogeneous polynomial in terms of vector components, such as

$$H(x, y, \bar{z}, \bar{r}, \bar{s}) = h^{ab}{}_{cde} x_b y_a s^e r^d z^c, \quad (\text{H.14})$$

as an *invariant* in  $V^q \otimes \bar{V}^p$  (repeated indices, as always, summed over). In this example, the coefficients  $h^{ab}{}_{cde}$  are components of invariant tensor  $h \in V^3 \otimes \bar{V}^2$ .

**Matrix group on vector space.** We will now apply these abstract group definitions to the set of  $[d \times d]$ -dimensional non-singular matrices  $\mathbf{A}, \mathbf{B}, \mathbf{C}, \dots \in GL(d)$  acting in a  $d$ -dimensional vector space  $V \in \mathbb{R}^d$ . The product of matrices  $\mathbf{A}$  and  $\mathbf{B}$  gives the matrix  $\mathbf{C}$ ,

$$\mathbf{C}x = \mathbf{B}(\mathbf{A}x) = (\mathbf{B}\mathbf{A})x \in V, \quad \forall x \in V.$$

The identity of the group is the unit matrix  $\mathbf{1}$  which leaves all vectors in  $V$  unchanged. Every matrix in the group has a unique inverse.

**Matrix representation of a group.** Let us now map the abstract group  $G$  *homeomorphically* on a group of matrices  $\mathbf{D}(G)$  acting on the vector space  $V$ , i.e., in such a way that the group properties, especially the group multiplication, are preserved:

1. Any  $g \in G$  is mapped to a matrix  $\mathbf{D}(g) \in \mathbf{D}(G)$ .
2. The group product  $g_2 \circ g_1 \in G$  is mapped onto the matrix product  $\mathbf{D}(g_2 \circ g_1) = \mathbf{D}(g_2)\mathbf{D}(g_1)$ .
3. The associativity is preserved:  $\mathbf{D}(g_3 \circ (g_2 \circ g_1)) = \mathbf{D}(g_3)(\mathbf{D}(g_2)\mathbf{D}(g_1)) = (\mathbf{D}(g_3)(\mathbf{D}(g_2))\mathbf{D}(g_1))$ .
4. The identity element  $e \in G$  is mapped onto the unit matrix  $\mathbf{D}(e) = \mathbf{1}$  and the inverse element  $g^{-1} \in G$  is mapped onto the inverse matrix  $\mathbf{D}(g^{-1}) = [\mathbf{D}(g)]^{-1} \equiv \mathbf{D}^{-1}(g)$ .

We call this matrix group  $\mathbf{D}(G)$  a linear or matrix *representation* of the group  $G$  in the *representation space*  $V$ . We emphasize here ‘*linear*’ in order to distinguish the matrix representations from other representations that do not have to be linear, in general. Throughout this appendix we only consider linear representations.

If the dimensionality of  $V$  is  $d$ , we say the representation is an *d-dimensional representation*. We will often abbreviate the notation by writing matrices  $\mathbf{D}(g) \in \mathbf{D}(G)$  as  $g$ , i.e.,  $x' = gx$  corresponds to the matrix operation  $x'_i = \sum_{j=1}^d \mathbf{D}(g)_{ij} x_j$ .

**Character of a representation.** The character of  $\chi_\alpha(g)$  of a  $d$ -dimensional representation  $\mathbf{D}(g)$  of the group element  $g \in G$  is defined as trace

$$\chi_\alpha(g) = \text{tr } \mathbf{D}(g) = \sum_{i=1}^d \mathbf{D}_{ii}(g).$$

Note that  $\chi(e) = d$ , since  $\mathbf{D}_{ij}(e) = \delta_{ij}$  for  $1 \leq i, j \leq d$ .

**Faithful representations, factor group.** If the mapping  $G$  on  $\mathbf{D}(G)$  is an isomorphism, the representation is said to be *faithful*. In this case the order of the group of matrices  $\mathbf{D}(G)$  is equal to the order  $|G|$  of the group. In general, however, there will be several elements  $h \in G$  that will be mapped on the unit matrix  $\mathbf{D}(h) = \mathbf{1}$ . This property can be used to define a subgroup  $H \subset G$  of the group  $G$  consisting of all elements  $h \in G$  that are mapped to the unit matrix of a given representation. Then the representation is a faithful representation of the *factor group*  $G/H$ .

**Equivalent representations, equivalence classes.** A representation of a group is by no means unique. If the basis in the  $d$ -dimensional vector space  $V$  is changed, the matrices  $\mathbf{D}(g)$  have to be replaced by their transformations  $\mathbf{D}'(g)$ , with the new matrices  $\mathbf{D}'(g)$  and the old matrices  $\mathbf{D}(g)$  are related by an equivalence transformation through a non-singular matrix  $\mathbf{C}$

$$\mathbf{D}'(g) = \mathbf{C} \mathbf{D}(g) \mathbf{C}^{-1}.$$

The group of matrices  $\mathbf{D}'(g)$  form a representation  $\mathbf{D}'(G)$  equivalent to the representation  $\mathbf{D}(G)$  of the group  $G$ . The equivalent representations have the same structure, although the matrices look different. Because of the cyclic nature of the trace the character of equivalent representations is the same

$$\chi(g) = \sum_{i=1}^n \mathbf{D}'_{ii}(g) = \text{tr } \mathbf{D}'(g) = \text{tr}(\mathbf{C} \mathbf{D}(g) \mathbf{C}^{-1}).$$

**Regular representation of a finite group.** The *regular* representation of a group is a special representation that is defined as follows: Combine the elements of a finite group into a vector  $\{g_1, g_2, \dots, g_{|G|}\}$ . Multiplication by any element  $g_\nu$  permutes  $\{g_1, g_2, \dots, g_{|G|}\}$  entries. We can represent the element  $g_\nu$  by the permutation it induces on the components of vector  $\{g_1, g_2, \dots, g_{|G|}\}$ . Thus for  $i, j = 1, \dots, |G|$ , we define the *regular representation*

$$\mathbf{D}_{ij}(g_\nu) = \begin{cases} \delta_{jl_i} & \text{if } g_\nu g_i = g_{l_i} \text{ with } l_i = 1, \dots, |G|, \\ 0 & \text{otherwise.} \end{cases}$$

In the regular representation the diagonal elements of all matrices are zero except for the identity element  $g_v = e$  with  $g_v g_i = g_i$ . So in the regular representation the character is given by

$$\chi(g) = \begin{cases} |G| & \text{for } g = e, \\ 0 & \text{for } g \neq e. \end{cases}$$

## H.2 Invariants and reducibility

What follows is a bit dry, so we start with a motivational quote from Hermann Weyl on the “so-called first main theorem of invariant theory”:

“All invariants are expressible in terms of a finite number among them. We cannot claim its validity for every group  $G$ ; rather, it will be our chief task to investigate for each particular group whether a finite integrity basis exists or not; the answer, to be sure, will turn out affirmative in the most important cases.”

It is easy to show that any rep of a finite group can be brought to unitary form, and the same is true of all compact Lie groups. Hence, in what follows, we specialize to unitary and hermitian matrices.

### H.2.1 Projection operators

For  $\mathbf{M}$  a hermitian matrix, there exists a diagonalizing unitary matrix  $\mathbf{C}$  such that

$$\mathbf{CMC}^\dagger = \begin{pmatrix} \boxed{\begin{matrix} \lambda_1 & \dots & 0 \\ & \ddots & \\ 0 & \dots & \lambda_1 \end{matrix}} & & 0 & & 0 \\ & & & & \\ & & \boxed{\begin{matrix} \lambda_2 & 0 & \dots & 0 \\ 0 & \lambda_2 & & \\ \vdots & & \ddots & \vdots \\ 0 & & \dots & \lambda_2 \end{matrix}} & & 0 \\ & & & & \\ 0 & & 0 & & \boxed{\begin{matrix} \lambda_3 & \dots \\ \vdots & \ddots \end{matrix}} \end{pmatrix}. \quad (\text{H.15})$$

Here  $\lambda_i \neq \lambda_j$  are the  $r$  distinct roots of the minimal *characteristic* (or *secular*) polynomial

$$\prod_{i=1}^r (\mathbf{M} - \lambda_i \mathbf{1}) = 0. \quad (\text{H.16})$$



In the matrix  $\mathbf{C}(\mathbf{M} - \lambda_2\mathbf{1})\mathbf{C}^\dagger$  the eigenvalues corresponding to  $\lambda_2$  are replaced by zeroes:

$$\left( \begin{array}{c|c|c} \lambda_1 - \lambda_2 & & \\ \hline & \lambda_1 - \lambda_2 & \\ \hline & & \begin{array}{c} 0 \\ \ddots \\ 0 \end{array} \\ \hline & & \lambda_3 - \lambda_2 \\ \hline & & \lambda_3 - \lambda_2 \\ & & \ddots \end{array} \right),$$

and so on, so the product over all factors  $(\mathbf{M} - \lambda_2\mathbf{1})(\mathbf{M} - \lambda_3\mathbf{1}) \dots$ , with exception of the  $(\mathbf{M} - \lambda_1\mathbf{1})$  factor, has nonzero entries only in the subspace associated with  $\lambda_1$ :

$$\mathbf{C} \prod_{j \neq 1} (\mathbf{M} - \lambda_j\mathbf{1}) \mathbf{C}^\dagger = \prod_{j \neq 1} (\lambda_1 - \lambda_j) \left( \begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 & 1 \\ \hline & & 0 \\ & & \begin{array}{c} 0 \\ 0 \\ 0 \\ \ddots \end{array} \end{array} \right).$$

Thus we can associate with each distinct root  $\lambda_i$  a *projection operator*  $\mathbf{P}_i$ ,

$$\mathbf{P}_i = \prod_{j \neq i} \frac{\mathbf{M} - \lambda_j\mathbf{1}}{\lambda_i - \lambda_j}, \quad (\text{H.17})$$

which acts as identity on the  $i$ th subspace, and zero elsewhere. For example, the projection operator onto the  $\lambda_1$  subspace is

$$\mathbf{P}_1 = \mathbf{C}^\dagger \left( \begin{array}{c|c|c} 1 & & \\ \hline & \ddots & \\ \hline & & 1 \\ \hline & & 0 \\ & & 0 \\ & & \ddots \\ & & 0 \end{array} \right) \mathbf{C}. \quad (\text{H.18})$$

The diagonalization matrix  $\mathbf{C}$  is deployed in the above only as a pedagogical device. The whole point of the projector operator formalism is that we *never* need to carry such explicit diagonalization; all we need are whatever invariant matrices  $\mathbf{M}$  we find convenient, the algebraic relations they satisfy, and orthonormality and completeness of  $\mathbf{P}_i$ : The matrices  $\mathbf{P}_i$  are *orthogonal*

$$\mathbf{P}_i \mathbf{P}_j = \delta_{ij} \mathbf{P}_j, \quad (\text{no sum on } j), \quad (\text{H.19})$$

and satisfy the *completeness relation*

$$\sum_{i=1}^r \mathbf{P}_i = \mathbf{1}. \quad (\text{H.20})$$

As  $\text{tr}(\mathbf{C}\mathbf{P}_i\mathbf{C}^\dagger) = \text{tr}\mathbf{P}_i$ , the dimension of the  $i$ th subspace is given by

$$d_i = \text{tr}\mathbf{P}_i. \quad (\text{H.21})$$

It follows from the characteristic equation (H.16) and the form of the projection operator (H.17) that  $\lambda_i$  is the eigenvalue of  $\mathbf{M}$  on  $\mathbf{P}_i$  subspace:

$$\mathbf{M}\mathbf{P}_i = \lambda_i\mathbf{P}_i, \quad (\text{no sum on } i). \quad (\text{H.22})$$

Hence, any matrix polynomial  $f(\mathbf{M})$  takes the scalar value  $f(\lambda_i)$  on the  $\mathbf{P}_i$  subspace

$$f(\mathbf{M})\mathbf{P}_i = f(\lambda_i)\mathbf{P}_i. \quad (\text{H.23})$$

This, of course, is the reason why one wants to work with irreducible reps: they reduce matrices and “operators” to pure numbers.

## H.2.2 Irreducible representations

Suppose there exist several linearly independent invariant [ $d \times d$ ] hermitian matrices  $\mathbf{M}_1, \mathbf{M}_2, \dots$ , and that we have used  $\mathbf{M}_1$  to decompose the  $d$ -dimensional vector space  $V = V_1 \oplus V_2 \oplus \dots$ . Can  $\mathbf{M}_2, \mathbf{M}_3, \dots$  be used to further decompose  $V_i$ ? Further decomposition is possible if, and only if, the invariant matrices commute:

$$[\mathbf{M}_1, \mathbf{M}_2] = 0, \quad (\text{H.24})$$

or, equivalently, if projection operators  $\mathbf{P}_j$  constructed from  $\mathbf{M}_2$  commute with projection operators  $\mathbf{P}_i$  constructed from  $\mathbf{M}_1$ ,

$$\mathbf{P}_i\mathbf{P}_j = \mathbf{P}_j\mathbf{P}_i. \quad (\text{H.25})$$

Usually the simplest choices of independent invariant matrices do not commute. In that case, the projection operators  $\mathbf{P}_i$  constructed from  $\mathbf{M}_1$  can be used to project commuting pieces of  $\mathbf{M}_2$ :

$$\mathbf{M}_2^{(i)} = \mathbf{P}_i\mathbf{M}_2\mathbf{P}_i, \quad (\text{no sum on } i).$$

That  $\mathbf{M}_2^{(i)}$  commutes with  $\mathbf{M}_1$  follows from the orthogonality of  $\mathbf{P}_i$ :

$$[\mathbf{M}_2^{(i)}, \mathbf{M}_1] = \sum_j \lambda_j [\mathbf{M}_2^{(i)}, \mathbf{P}_j] = 0. \quad (\text{H.26})$$

Now the characteristic equation for  $\mathbf{M}_2^{(i)}$  (if nontrivial) can be used to decompose  $V_i$  subspace.

An invariant matrix  $\mathbf{M}$  induces a decomposition only if its diagonalized form (H.15) has more than one distinct eigenvalue; otherwise it is proportional to the unit matrix and commutes trivially with all group elements. A rep is said to be *irreducible* if all invariant matrices that can be constructed are proportional to the unit matrix.

According to (H.13), an invariant matrix  $\mathbf{M}$  commutes with group transformations  $[G, \mathbf{M}] = 0$ . Projection operators (H.17) constructed from  $\mathbf{M}$  are polynomials in  $\mathbf{M}$ , so they also commute with all  $g \in \mathcal{G}$ :

$$[G, \mathbf{P}_i] = 0 \quad (\text{H.27})$$

Hence, a  $[d \times d]$  matrix rep can be written as a direct sum of  $[d_i \times d_i]$  matrix reps:

$$G = \mathbf{1}G\mathbf{1} = \sum_{i,j} \mathbf{P}_i G \mathbf{P}_j = \sum_i \mathbf{P}_i G \mathbf{P}_i = \sum_i G_i. \quad (\text{H.28})$$

In the diagonalized rep (H.18), the matrix  $\mathbf{g}$  has a block diagonal form:

$$\mathbf{C}\mathbf{g}\mathbf{C}^\dagger = \begin{bmatrix} \mathbf{g}_1 & 0 & 0 \\ 0 & \mathbf{g}_2 & 0 \\ 0 & 0 & \ddots \end{bmatrix}, \quad \mathbf{g} = \sum_i \mathbf{C}^i \mathbf{g}_i \mathbf{C}_i. \quad (\text{H.29})$$

The rep  $\mathbf{g}_i$  acts only on the  $d_i$ -dimensional subspace  $V_i$  consisting of vectors  $\mathbf{P}_i q$ ,  $q \in V$ . In this way an invariant  $[d \times d]$  hermitian matrix  $\mathbf{M}$  with  $r$  distinct eigenvalues induces a decomposition of a  $d$ -dimensional vector space  $V$  into a direct sum of  $d_i$ -dimensional vector subspaces  $V_i$ :

$$V \xrightarrow{\mathbf{M}} V_1 \oplus V_2 \oplus \dots \oplus V_r. \quad (\text{H.30})$$

### H.3 Lattice derivatives

Consider a smooth function  $\phi(x)$  evaluated on a finite  $d$ -dimensional lattice

$$\phi_\ell = \phi(x), \quad x = a\ell = \text{lattice point}, \quad \ell \in \mathbf{Z}^d, \quad (\text{H.31})$$

where  $a$  is the lattice spacing and there are  $N^d$  points in all. A vector  $\phi$  specifies a lattice configuration. Assume the lattice is hyper-cubic, and let  $\hat{n}_\mu \in \{\hat{n}_1, \hat{n}_2, \dots, \hat{n}_d\}$  be the unit lattice cell vectors pointing along the  $d$  positive directions,  $|\hat{n}_\mu| = 1$ . The *lattice partial derivative* is then

$$(\partial_\mu \phi)_\ell = \frac{\phi(x + a\hat{n}_\mu) - \phi(x)}{a} = \frac{\phi_{\ell+\hat{n}_\mu} - \phi_\ell}{a}.$$

Anything else with the correct  $a \rightarrow 0$  limit would do, but this is the simplest choice. We can rewrite the derivative as a linear operator, by introducing the *hopping operator* (or “shift,” or “step”) in the direction  $\mu$

$$(\mathbf{h}_\mu)_{\ell j} = \delta_{\ell+\hat{n}_\mu, j}. \quad (\text{H.32})$$

As  $\mathbf{h}$  will play a central role in what follows, it pays to understand what it does, so we write it out for the 1-dimensional case in its full  $[N \times N]$  matrix glory:

$$\mathbf{h} = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & \ddots & \ddots \\ & & & & 0 & 1 \\ 1 & & & & & 0 \end{pmatrix}. \quad (\text{H.33})$$

We will assume throughout that the lattice is *periodic* in each  $\hat{n}_\mu$  direction; this is the easiest boundary condition to work with if we are interested in large lattices where surface effects are negligible.

Applied on the lattice configuration  $\phi = (\phi_1, \phi_2, \dots, \phi_N)$ , the hopping operator shifts the lattice by one site,  $\mathbf{h}\phi = (\phi_2, \phi_3, \dots, \phi_N, \phi_1)$ . Its transpose shifts the entries the other way, so the transpose is also the inverse

$$\mathbf{h}^{-1} = \mathbf{h}^T. \quad (\text{H.34})$$

The lattice derivative can now be written as a multiplication by a matrix:

$$\partial_\mu \phi_\ell = \frac{1}{a} (\mathbf{h}_\mu - \mathbf{1})_{\ell j} \phi_j.$$

In the 1-dimensional case the  $[N \times N]$  matrix representation of the lattice derivative is:

$$\partial = \frac{1}{a} \begin{pmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & & \ddots & \ddots \\ & & & & 1 & -1 \\ 1 & & & & & -1 \end{pmatrix}. \quad (\text{H.35})$$

To belabor the obvious: On a finite lattice of  $N$  points a derivative is simply a finite  $[N \times N]$  matrix. Continuum field theory is a world in which the lattice is so fine that it looks smooth to us. Whenever someone calls something an “operator,” think “matrix.” For finite-dimensional spaces a linear operator *is* a matrix; things get subtler for infinite-dimensional spaces.

### H.3.1 Lattice Laplacian

In the continuum, integration by parts moves  $\partial$  around; on a lattice this amounts to a matrix transposition

$$\left[ (\mathbf{h}_\mu - \mathbf{1}) \phi \right]^T \cdot \left[ (\mathbf{h}_\mu - \mathbf{1}) \phi \right] = \phi^T \cdot (\mathbf{h}_\mu^{-1} - \mathbf{1}) (\mathbf{h}_\mu - \mathbf{1}) \cdot \phi.$$

If you are wondering where the “integration by parts” minus sign is, it is there in discrete case as well. It comes from the identity  $\partial^T = -\mathbf{h}^{-1}\partial$ . The combination  $\Delta = \mathbf{h}^{-1}\partial^2$

$$\Delta = -\frac{1}{a^2} \sum_{\mu=1}^d (\mathbf{h}_\mu^{-1} - \mathbf{1}) (\mathbf{h}_\mu - \mathbf{1}) = -\frac{2}{a^2} \sum_{\mu=1}^d \left( \mathbf{1} - \frac{1}{2}(\mathbf{h}_\mu^{-1} + \mathbf{h}_\mu) \right) \quad (\text{H.36})$$

is the *lattice Laplacian*. We shall show below that this Laplacian has the correct continuum limit. It is the simplest spatial derivative allowed for  $x \rightarrow -x$  symmetric actions. In the 1-dimensional case the  $[N \times N]$  matrix representation of the lattice Laplacian is:

$$\Delta = \frac{1}{a^2} \begin{pmatrix} -2 & 1 & & & 1 \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & 1 & \ddots & \\ & & & & 1 \\ 1 & & & & 1 & -2 \end{pmatrix}. \quad (\text{H.37})$$

The lattice Laplacian measures the second variation of a field  $\phi_\ell$  across three neighboring sites. You can easily check that it does what the second derivative is supposed to do by applying it to a parabola restricted to the lattice,  $\phi_\ell = \phi(\ell)$ , where  $\phi(\ell)$  is defined by the value of the continuum function  $\phi(x) = x^2$  at the lattice point  $\ell$ .

### H.3.2 Inverting the Laplacian

Evaluation of perturbative corrections in (28.21) requires that we come to grips with the “free” or “bare” propagator  $M$ . While the the Laplacian is a simple

difference operator (H.37), its inverse is a messier object. A way to compute is to start expanding the propagator  $M$  as a power series in the Laplacian

$$M = \frac{1}{m^2 - \Delta} = \frac{1}{m^2} \sum_{k=0}^{\infty} \frac{1}{m^{2k}} \Delta^k. \quad (\text{H.38})$$

As  $\Delta$  is a finite matrix, the expansion is convergent for sufficiently large  $m^2$ . To get a feeling for what is involved in evaluating such series, evaluate  $\Delta^2$  in the 1-dimensional case:

$$\Delta^2 = \frac{1}{a^4} \begin{pmatrix} 6 & -4 & 1 & & & & & & & 1 & -4 \\ -4 & 6 & -4 & 1 & & & & & & & & \\ 1 & -4 & 6 & -4 & 1 & & & & & & & \\ & & 1 & -4 & \ddots & & & & & & & \\ & & & & & & & & & & 6 & -4 \\ -4 & 1 & & & & & & & & & 1 & -4 & 6 \end{pmatrix}. \quad (\text{H.39})$$

What  $\Delta^3$ ,  $\Delta^4$ ,  $\dots$  contributions look like is now clear; as we include higher and higher powers of the Laplacian, the propagator matrix fills up; while the *inverse* propagator is differential operator connecting only the nearest neighbors, the propagator is integral operator, connecting every lattice site to any other lattice site.

These matrices can be evaluated as is, on the lattice, and sometime it is evaluated this way, but in case at hand a wonderful simplification follows from the observation that the lattice action is translationally invariant. We will show how this works in sect. H.4.

## H.4 Periodic lattices

Our task now is to transform  $M$  into a form suitable to explicit evaluation. Consider the effect of a  $\phi \rightarrow \mathbf{h}\phi$  translation on the matrix polynomial

$$S[\mathbf{h}\phi] = -\frac{1}{2} \phi^T \cdot \mathbf{h}^T M^{-1} \mathbf{h} \cdot \phi.$$

As  $M^{-1}$  is constructed from  $\mathbf{h}$  and its inverse,  $M^{-1}$  and  $\mathbf{h}$  commute, and the function  $S[\mathbf{h}\phi]$  is  $\mathbf{h}$ -invariant,

$$S[\mathbf{h}\phi] = S[\phi] = -\frac{1}{2} \phi^T \cdot M^{-1} \cdot \phi. \quad (\text{H.40})$$

If a function (in this case, the function  $S[\phi]$ ) defined on a vector space (in this case, the configuration  $\phi$ ) commutes with a linear operator  $\mathbf{h}$ , then the eigenvalues

of  $\mathbf{h}$  can be used to decompose the  $\phi$  vector space into invariant subspaces. For a hyper-cubic lattice the translations in different directions commute,  $\mathbf{h}_\mu \mathbf{h}_\nu = \mathbf{h}_\nu \mathbf{h}_\mu$ , so it is sufficient to understand the spectrum of the 1-dimensional shift operator (H.33). To develop a feeling for how this reduction to invariant subspaces works in practice, let us continue humbly, by expanding the scope of our deliberations to a lattice consisting of 2 points.

#### H.4.1 A 2-point lattice diagonalized

The action of the shift operator  $\mathbf{h}$  (H.33) on a 2-point lattice  $\phi = (\phi_1, \phi_2)$  is to permute the two lattice sites

$$\mathbf{h} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

As exchange repeated twice brings us back to the original configuration,  $\mathbf{h}^2 = \mathbf{1}$ , and the characteristic polynomial of  $\mathbf{h}$  is

$$(\mathbf{h} + \mathbf{1})(\mathbf{h} - \mathbf{1}) = 0,$$

with eigenvalues  $\lambda_0 = 1, \lambda_1 = -1$ . Construct now the symmetrization, antisymmetrization projection operators

$$P_0 = \frac{\mathbf{h} - \lambda_1 \mathbf{1}}{\lambda_0 - \lambda_1} = \frac{1}{2}(\mathbf{1} + \mathbf{h}) = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad (\text{H.41})$$

$$P_1 = \frac{\mathbf{h} - \lambda_0 \mathbf{1}}{\lambda_1 - \lambda_0} = \frac{1}{2}(\mathbf{1} - \mathbf{h}) = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \quad (\text{H.42})$$

Noting that  $P_0 + P_1 = \mathbf{1}$ , we can project the lattice configuration  $\phi$  onto the two eigenvectors of  $\mathbf{h}$ :

$$\begin{aligned} \phi &= \mathbf{1} \phi = P_0 \cdot \phi + P_1 \cdot \phi, \\ \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} &= \frac{(\phi_1 + \phi_2)}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{(\phi_1 - \phi_2)}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{aligned} \quad (\text{H.43})$$

$$= \tilde{\phi}_0 \hat{n}_0 + \tilde{\phi}_1 \hat{n}_1. \quad (\text{H.44})$$

As  $P_0 P_1 = 0$ , the symmetric and the antisymmetric configurations transform separately under any linear transformation constructed from  $\mathbf{h}$  and its powers.

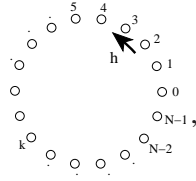
In this way the characteristic equation  $\mathbf{h}^2 = \mathbf{1}$  enables us to reduce the 2-dimensional lattice configuration to two 1-dimensional ones, on which the value of the shift operator (shift matrix)  $\mathbf{h}$  is a number,  $\lambda \in \{1, -1\}$ , and the eigenvectors are  $\hat{n}_0 = \frac{1}{\sqrt{2}}(1, 1)$ ,  $\hat{n}_1 = \frac{1}{\sqrt{2}}(1, -1)$ . We have inserted  $\sqrt{2}$  factors only for convenience, in order that the eigenvectors be normalized unit vectors. As we shall now see,  $(\tilde{\phi}_0, \tilde{\phi}_1)$  is the 2-site periodic lattice discrete Fourier transform of the field  $(\phi_1, \phi_2)$ .

## H.5 Discrete Fourier transforms

Now let us generalize this reduction to a 1-dimensional periodic lattice with  $N$  sites.

Each application of  $\mathbf{h}$  translates the lattice one step; in  $N$  steps the lattice is back in the original configuration

$$\mathbf{h}^N = \mathbf{1}$$



so the eigenvalues of  $\mathbf{h}$  are the  $N$  distinct  $N$ -th roots of unity

$$\mathbf{h}^N - \mathbf{1} = \prod_{k=0}^{N-1} (\mathbf{h} - \omega^k \mathbf{1}) = 0, \quad \omega = e^{i\frac{2\pi}{N}}. \quad (\text{H.45})$$

As the eigenvalues are all distinct and  $N$  in number, the space is decomposed into  $N$  1-dimensional subspaces. The general theory (expounded in appendix H.2) associates with the  $k$ -th eigenvalue of  $\mathbf{h}$  a projection operator that projects a configuration  $\phi$  onto  $k$ -th eigenvector of  $\mathbf{h}$ ,

$$P_k = \prod_{j \neq k} \frac{\mathbf{h} - \lambda_j \mathbf{1}}{\lambda_k - \lambda_j}. \quad (\text{H.46})$$

A factor  $(\mathbf{h} - \lambda_j \mathbf{1})$  kills the  $j$ -th eigenvector  $\varphi_j$  component of an arbitrary vector in expansion  $\phi = \dots + \tilde{\phi}_j \varphi_j + \dots$ . The above product kills everything but the eigen-direction  $\varphi_k$ , and the factor  $\prod_{j \neq k} (\lambda_k - \lambda_j)$  ensures that  $P_k$  is normalized as a projection operator. The set of the projection operators is complete

$$\sum_k P_k = \mathbf{1} \quad (\text{H.47})$$

and orthonormal

$$P_k P_j = \delta_{kj} P_k \quad (\text{no sum on } k). \quad (\text{H.48})$$

Constructing explicit eigenvectors is usually not the best way to fritter one's youth away, as choice of basis is largely arbitrary, and all of the content of the theory is in projection operators. However, in case at hand the eigenvectors are so simple that we can forget the general theory, and construct the solutions of the eigenvalue condition

$$\mathbf{h} \varphi_k = \omega^k \varphi_k \quad (\text{H.49})$$



by hand:

$$\frac{1}{\sqrt{N}} \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & \ddots & \\ & & & & 0 & 1 \\ 1 & & & & & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \omega^k \\ \omega^{2k} \\ \omega^{3k} \\ \vdots \\ \omega^{(N-1)k} \end{pmatrix} = \omega^k \frac{1}{\sqrt{N}} \begin{pmatrix} 1 \\ \omega^k \\ \omega^{2k} \\ \omega^{3k} \\ \vdots \\ \omega^{(N-1)k} \end{pmatrix}$$

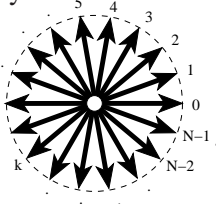
The  $1/\sqrt{N}$  factor is chosen in order that  $\varphi_k$  be normalized unit vectors

$$\begin{aligned} \varphi_k^\dagger \cdot \varphi_k &= \frac{1}{N} \sum_{k=0}^{N-1} 1 = 1, \quad (\text{no sum on } k) \\ \varphi_k^\dagger &= \frac{1}{\sqrt{N}} (1, \omega^{-k}, \omega^{-2k}, \dots, \omega^{-(N-1)k}). \end{aligned} \quad (\text{H.50})$$

The eigenvectors are orthonormal

$$\varphi_k^\dagger \cdot \varphi_j = \delta_{kj}, \quad (\text{H.51})$$

as the explicit evaluation of  $\varphi_k^\dagger \cdot \varphi_j$  yields the *Kronecker delta function for a periodic lattice*

$$\delta_{kj} = \frac{1}{N} \sum_{\ell=0}^{N-1} e^{i\frac{2\pi}{N}(k-j)\ell} \quad (\text{H.52})$$


The sum is over the  $N$  unit vectors pointing at a uniform distribution of points on the complex unit circle; they cancel each other unless  $k = j \pmod{N}$ , in which case each term in the sum equals 1.

The projection operators can be expressed in terms of the eigenvectors (H.49), (H.50) as

$$(P_k)_{\ell\ell'} = (\varphi_k)_\ell (\varphi_k^\dagger)_{\ell'} = \frac{1}{N} e^{i\frac{2\pi}{N}(\ell-\ell')k}, \quad (\text{no sum on } k). \quad (\text{H.53})$$

The completeness (H.47) follows from (H.52), and the orthonormality (H.48) from (H.51).

$\tilde{\phi}_k$ , the projection of the  $\phi$  configuration on the  $k$ -th subspace is given by

$$\begin{aligned} (P_k \cdot \phi)_\ell &= \tilde{\phi}_k (\varphi_k)_\ell, \quad (\text{no sum on } k) \\ \tilde{\phi}_k &= \varphi_k^\dagger \cdot \phi = \frac{1}{\sqrt{N}} \sum_{\ell=0}^{N-1} e^{-i\frac{2\pi}{N}k\ell} \phi_\ell \end{aligned} \quad (\text{H.54})$$

We recognize  $\tilde{\phi}_k$  as the *discrete Fourier transform* of  $\phi_\ell$ . Hopefully rediscovering it this way helps you a little toward understanding why Fourier transforms are full of  $e^{ix \cdot p}$  factors (they are eigenvalues of the generator of translations) and when are they the natural set of basis functions (only if the theory is translationally invariant).

### H.5.1 Fourier transform of the propagator

Now insert the identity  $\sum P_k = \mathbf{1}$  wherever profitable:

$$\mathbf{M} = \mathbf{1M1} = \sum_{kk'} P_k \mathbf{M} P_{k'} = \sum_{kk'} \varphi_k (\varphi_k^\dagger \cdot \mathbf{M} \cdot \varphi_{k'}) \varphi_{k'}^\dagger.$$

The matrix

$$\tilde{M}_{kk'} = (\varphi_k^\dagger \cdot \mathbf{M} \cdot \varphi_{k'}) \quad (\text{H.55})$$

is the Fourier space representation of  $\mathbf{M}$ . According to (H.51) the matrix  $U_{k\ell} = (\varphi_k)_\ell = \frac{1}{\sqrt{N}} e^{i\frac{2\pi}{N}k\ell}$  is a unitary matrix, and the Fourier transform is a linear, unitary transformation  $UU^\dagger = \sum P_k = \mathbf{1}$  with Jacobian  $\det U = 1$ . The form of the invariant function (H.40) does not change under  $\phi \rightarrow \tilde{\phi}_k$  transformation, and from the formal point of view, it does not matter whether we compute in the Fourier space or in the configuration space that we started out with. For example, the trace of  $\mathbf{M}$  is the trace in either representation

$$\begin{aligned} \text{tr } \mathbf{M} &= \sum_{\ell} M_{\ell\ell} = \sum_{kk'} \sum_{\ell} (P_k \mathbf{M} P_{k'})_{\ell\ell} \\ &= \sum_{kk'} \sum_{\ell} (\varphi_k)_\ell (\varphi_k^\dagger \cdot \mathbf{M} \cdot \varphi_{k'}) (\varphi_{k'})_\ell = \sum_{kk'} \delta_{kk'} \tilde{M}_{kk'} = \text{tr } \tilde{\mathbf{M}}. \end{aligned}$$

From this it follows that  $\text{tr } \mathbf{M}^n = \text{tr } \tilde{\mathbf{M}}^n$ , and from the  $\text{tr } \ln = \ln \text{tr}$  relation that  $\det \mathbf{M} = \det \tilde{\mathbf{M}}$ . In fact, any scalar combination of  $\phi$ 's,  $J$ 's and couplings, such as the partition function  $Z[J]$ , has exactly the same form in the configuration and the Fourier space.

OK, a dizzying quantity of indices. But what's the pay-back?

### H.5.2 Lattice Laplacian diagonalized

Now use the eigenvalue equation (H.49) to convert  $\mathbf{h}$  matrices into scalars. If  $\mathbf{M}$  commutes with  $\mathbf{h}$ , then  $(\varphi_k^\dagger \cdot \mathbf{M} \cdot \varphi_{k'}) = \tilde{M}_k \delta_{kk'}$ , and the matrix  $\mathbf{M}$  acts as a multiplication by the scalar  $\tilde{M}_k$  on the  $k$ -th subspace. For example, for the

1-dimensional version of the lattice Laplacian (H.36) the projection on the  $k$ -th subspace is

$$\begin{aligned} (\varphi_k^\dagger \cdot \Delta \cdot \varphi_{k'}) &= \frac{2}{a^2} \left( \frac{1}{2} (\omega^{-k} + \omega^k) - 1 \right) (\varphi_k^\dagger \cdot \varphi_{k'}) \\ &= \frac{2}{a^2} \left( \cos \left( \frac{2\pi}{N} k \right) - 1 \right) \delta_{kk'} \end{aligned} \quad (\text{H.56})$$

In the  $k$ -th subspace the bare propagator (H.56) is simply a number, and, in contrast to the mess generated by (H.38), there is nothing to inverting  $M^{-1}$ :

$$(\varphi_{\mathbf{k}}^\dagger \cdot M \cdot \varphi_{\mathbf{k}'}) = (\tilde{G}_0)_{\mathbf{k}} \delta_{\mathbf{k}\mathbf{k}'} = \frac{1}{\beta m_0'^2 - \frac{2c}{a^2} \sum_{\mu=1}^d \left( \cos \left( \frac{2\pi}{N} k_\mu \right) - 1 \right)}, \quad (\text{H.57})$$

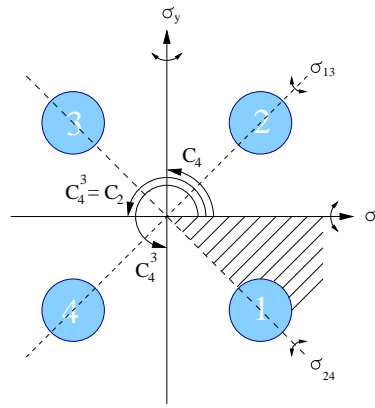
where  $\mathbf{k} = (k_1, k_2, \dots, k_\mu)$  is a  $d$ -dimensional vector in the  $N^d$ -dimensional dual lattice.

Going back to the partition function (28.21) and sticking in the factors of  $\mathbf{1}$  into the bilinear part of the interaction, we replace the spatial  $J_\ell$  by its Fourier transform  $\tilde{J}_k$ , and the spatial propagator  $(M)_{\ell\ell'}$  by the diagonalized Fourier transformed  $(\tilde{G}_0)_k$

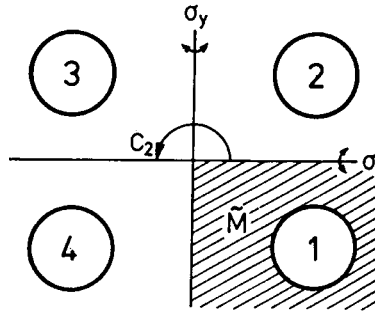
$$J^T \cdot M \cdot J = \sum_{k,k'} (J^T \cdot \varphi_k) (\varphi_k^\dagger \cdot M \cdot \varphi_{k'}) (\varphi_{k'}^\dagger \cdot J) = \sum_k \tilde{J}_k^\dagger (\tilde{G}_0)_k \tilde{J}_k. \quad (\text{H.58})$$

## H.6 $C_{4v}$ factorization

If an  $N$ -disk arrangement has  $C_N$  symmetry, and the disk visitation sequence is given by disk labels  $\{\epsilon_1 \epsilon_2 \epsilon_3 \dots\}$ , only the relative increments  $\rho_i = \epsilon_{i+1} - \epsilon_i \bmod N$  matter. Symmetries under reflections across axes increase the group to  $C_{Nv}$ , and add relations between symbols:  $\{\epsilon_i\}$  and  $\{N - \epsilon_i\}$  differ only by a reflection. As a consequence of this reflection increments become decrements until the next reflection and vice versa. Consider four equal disks placed on the vertices of a square (figure H.1). The symmetry group consists of the identity  $\mathbf{e}$ , the two reflections  $\sigma_x, \sigma_y$  across  $x, y$  axes, the two diagonal reflections  $\sigma_{13}, \sigma_{24}$ , and the three rotations  $C_4, C_2$  and  $C_4^3$  by angles  $\pi/2, \pi$  and  $3\pi/2$ . We start by exploiting the  $C_4$  subgroup symmetry in order to replace the absolute labels  $\epsilon_i \in \{1, 2, 3, 4\}$  by relative increments  $\rho_i \in \{1, 2, 3\}$ . By reflection across diagonals, an increment by 3 is equivalent to an increment by 1 and a reflection; this new symbol will be called  $\underline{1}$ . Our convention will be to first perform the increment and then to change the orientation due to the reflection. As an example, consider the fundamental domain cycle 112. Taking the disk  $1 \rightarrow$  disk 2 segment as the starting segment, this symbol string is mapped into the disk visitation sequence  $1_{+1} 2_{+1} 3_{+2} 1 \dots = \underline{123}$ , where the subscript indicates the increments (or decrements) between neighboring symbols;



**Figure H.1:** Symmetries of four disks on a square. A fundamental domain indicated by the shaded wedge.



**Figure H.2:** Symmetries of four disks on a rectangle. A fundamental domain indicated by the shaded wedge.

the period of the cycle  $\overline{112}$  is thus 3 in both the fundamental domain and the full space. Similarly, the cycle  $\underline{112}$  will be mapped into  $1_{+1}2_{-1}1_{-2}3_{-1}2_{+1}3_{+2}1 = 121323$  (note that the fundamental domain symbol  $\underline{1}$  corresponds to a flip in orientation after the second and fifth symbols); this time the period in the full space is twice that of the fundamental domain. In particular, the fundamental domain fixed points correspond to the following 4-disk cycles:

4-disk		reduced
12	$\leftrightarrow$	$\underline{1}$
1234	$\leftrightarrow$	$\underline{1}$
13	$\leftrightarrow$	2

Conversions for all periodic orbits of reduced symbol period less than 5 are listed in table H.1.

This symbolic dynamics is closely related to the group-theoretic structure of the dynamics: the global 4-disk trajectory can be generated by mapping the fundamental domain trajectories onto the full 4-disk space by the accumulated product of the  $C_{4v}$  group elements  $g_1 = C$ ,  $g_2 = C^2$ ,  $g_{\underline{1}} = \sigma_{diag}C = \sigma_{axis}$ , where  $C$  is a rotation by  $\pi/2$ . In the  $\underline{112}$  example worked out above, this yields  $g_{\underline{1}12} = g_2g_1g_{\underline{1}} = C^2C\sigma_{axis} = \sigma_{diag}$ , listed in the last column of table H.1. Our convention is to multiply group elements in the reverse order with respect to the symbol sequence. We need these group elements for our next step, the dynamical zeta function factorizations.

**Table H.1:**  $C_{4v}$  correspondence between the ternary fundamental domain prime cycles  $\tilde{p}$  and the full 4-disk  $\{1,2,3,4\}$  labeled cycles  $p$ , together with the  $C_{4v}$  transformation that maps the end point of the  $\tilde{p}$  cycle into an irreducible segment of the  $p$  cycle. For typographical convenience, the symbol  $\underline{1}$  of sect. H.6 has been replaced by 0, so that the ternary alphabet is  $\{0, 1, 2\}$ . The degeneracy of the  $p$  cycle is  $m_p = 8n_{\tilde{p}}/n_p$ . Orbit  $\bar{2}$  is the sole boundary orbit, invariant both under a rotation by  $\pi$  and a reflection across a diagonal. The two pairs of cycles marked by (a) and (b) are related by time reversal, but cannot be mapped into each other by  $C_{4v}$  transformations.

$\tilde{p}$	$p$	$\mathbf{h}_{\tilde{p}}$	$\tilde{p}$	$p$	$\mathbf{h}_{\tilde{p}}$
0	12	$\sigma_x$	0001	1212 1414	$\sigma_{24}$
1	1 2 3 4	$C_4$	0002	1212 4343	$\sigma_y$
2	13	$C_2, \sigma_{13}$	0011	1212 3434	$C_2$
01	12 14	$\sigma_{24}$	0012	1212 4141 3434 2323	$C_4^3$
02	12 43	$\sigma_y$	0021 (a)	1213 4142 3431 2324	$C_4^3$
12	12 41 34 23	$C_4^3$	0022	1213	$e$
001	121 232 343 414	$C_4$	0102 (a)	1214 2321 3432 4143	$C_4$
002	121 343	$C_2$	0111	1214 3234	$\sigma_{13}$
011	121 434	$\sigma_y$	0112 (b)	1214 2123	$\sigma_x$
012	121 323	$\sigma_{13}$	0121 (b)	1213 2124	$\sigma_x$
021	124 324	$\sigma_{13}$	0122	1213 1413	$\sigma_{24}$
022	124 213	$\sigma_x$	0211	1243 2134	$\sigma_x$
112	123	$e$	0212	1243 1423	$\sigma_{24}$
122	124 231 342 413	$C_4$	0221	1242 1424	$\sigma_{24}$
			0222	1242 4313	$\sigma_y$
			1112	1234 2341 3412 4123	$C_4$
			1122	1231 3413	$C_2$
			1222	1242 4131 3424 2313	$C_4^3$

The  $C_{4v}$  group has four 1-dimensional representations, either symmetric ( $A_1$ ) or antisymmetric ( $A_2$ ) under both types of reflections, or symmetric under one and antisymmetric under the other ( $B_1, B_2$ ), and a degenerate pair of 2-dimensional representations  $E$ . Substituting the  $C_{4v}$  characters

$C_{4v}$	$A_1$	$A_2$	$B_1$	$B_2$	$E$
$e$	1	1	1	1	2
$C_2$	1	1	1	1	-2
$C_4, C_4^3$	1	1	-1	-1	0
$\sigma_{axes}$	1	-1	1	-1	0
$\sigma_{diag}$	1	-1	-1	1	0

into (21.15) we obtain:

$$\begin{array}{l}
 \mathbf{h}_{\tilde{p}} \\
 e: (1 - t_{\tilde{p}})^8 = (1 - t_{\tilde{p}}) (1 - t_{\tilde{p}}) (1 - t_{\tilde{p}}) (1 - t_{\tilde{p}}) (1 - t_{\tilde{p}}) (1 - t_{\tilde{p}}) (1 - t_{\tilde{p}}) (1 - t_{\tilde{p}}) \\
 C_2: (1 - t_{\tilde{p}}^2)^4 = (1 - t_{\tilde{p}}) (1 - t_{\tilde{p}}) (1 - t_{\tilde{p}}) (1 - t_{\tilde{p}}) (1 - t_{\tilde{p}}) (1 - t_{\tilde{p}}) (1 - t_{\tilde{p}}) (1 - t_{\tilde{p}}) \\
 C_4, C_4^3: (1 - t_{\tilde{p}}^4)^2 = (1 - t_{\tilde{p}}) (1 - t_{\tilde{p}}) (1 + t_{\tilde{p}}) (1 + t_{\tilde{p}}) (1 + t_{\tilde{p}}^2)^2 \\
 \sigma_{axes}: (1 - t_{\tilde{p}}^2)^4 = (1 - t_{\tilde{p}}) (1 + t_{\tilde{p}}) (1 - t_{\tilde{p}}) (1 + t_{\tilde{p}}) (1 - t_{\tilde{p}}^2)^2 \\
 \sigma_{diag}: (1 - t_{\tilde{p}}^2)^4 = (1 - t_{\tilde{p}}) (1 + t_{\tilde{p}}) (1 + t_{\tilde{p}}) (1 - t_{\tilde{p}}) (1 - t_{\tilde{p}}^2)^2
 \end{array}$$

The possible irreducible segment group elements  $\mathbf{h}_{\tilde{p}}$  are listed in the first column;  $\sigma_{axes}$  denotes a reflection across either the x-axis or the y-axis, and  $\sigma_{diag}$  denotes a reflection across a diagonal (see figure H.1). In addition, degenerate pairs of boundary orbits can run along the symmetry lines in the full space, with the fundamental domain group theory weights  $\mathbf{h}_p = (C_2 + \sigma_x)/2$  (axes) and  $\mathbf{h}_p = (C_2 + \sigma_{13})/2$  (diagonals) respectively:

$$\begin{array}{rccccc}
 & & A_1 & A_2 & B_1 & B_2 & E \\
 \text{axes:} & (1 - t_{\tilde{p}}^2)^2 & = & (1 - t_{\tilde{p}})(1 - 0t_{\tilde{p}})(1 - t_{\tilde{p}})(1 - 0t_{\tilde{p}})(1 + t_{\tilde{p}})^2 \\
 \text{diagonals:} & (1 - t_{\tilde{p}}^2)^2 & = & (1 - t_{\tilde{p}})(1 - 0t_{\tilde{p}})(1 - 0t_{\tilde{p}})(1 - t_{\tilde{p}})(1 + t_{\tilde{p}})^2 \text{ (H.59)}
 \end{array}$$

(we have assumed that  $t_{\tilde{p}}$  does not change sign under reflections across symmetry axes). For the 4-disk arrangement considered here only the diagonal orbits  $\overline{13}, \overline{24}$  occur; they correspond to the  $\overline{2}$  fixed point in the fundamental domain.

The  $A_1$  subspace in  $C_{4v}$  cycle expansion is given by

$$\begin{aligned}
 1/\zeta_{A_1} &= (1 - t_0)(1 - t_1)(1 - t_2)(1 - t_{01})(1 - t_{02})(1 - t_{12}) \\
 &\quad (1 - t_{001})(1 - t_{002})(1 - t_{011})(1 - t_{012})(1 - t_{021})(1 - t_{022})(1 - t_{112}) \\
 &\quad (1 - t_{122})(1 - t_{0001})(1 - t_{0002})(1 - t_{0011})(1 - t_{0012})(1 - t_{0021}) \dots \\
 &= 1 - t_0 - t_1 - t_2 - (t_{01} - t_0t_1) - (t_{02} - t_0t_2) - (t_{12} - t_1t_2) \\
 &\quad - (t_{001} - t_0t_{01}) - (t_{002} - t_0t_{02}) - (t_{011} - t_1t_{01}) \\
 &\quad - (t_{022} - t_2t_{02}) - (t_{112} - t_1t_{12}) - (t_{122} - t_2t_{12}) \\
 &\quad - (t_{012} + t_{021} + t_0t_1t_2 - t_0t_{12} - t_1t_{02} - t_2t_{01}) \dots \quad \text{(H.60)}
 \end{aligned}$$

(for typographical convenience,  $\underline{1}$  is replaced by 0 in the remainder of this section). For 1-dimensional representations, the characters can be read off the symbol strings:  $\chi_{A_2}(\mathbf{h}_{\tilde{p}}) = (-1)^{n_0}$ ,  $\chi_{B_1}(\mathbf{h}_{\tilde{p}}) = (-1)^{n_1}$ ,  $\chi_{B_2}(\mathbf{h}_{\tilde{p}}) = (-1)^{n_0+n_1}$ , where  $n_0$  and  $n_1$  are the number of times symbols 0, 1 appear in the  $\tilde{p}$  symbol string. For  $B_2$  all  $t_p$  with an odd total number of 0's and 1's change sign:

$$\begin{aligned}
 1/\zeta_{B_2} &= (1 + t_0)(1 + t_1)(1 - t_2)(1 - t_{01})(1 + t_{02})(1 + t_{12}) \\
 &\quad (1 + t_{001})(1 - t_{002})(1 + t_{011})(1 - t_{012})(1 - t_{021})(1 + t_{022})(1 - t_{112}) \\
 &\quad (1 + t_{122})(1 - t_{0001})(1 + t_{0002})(1 - t_{0011})(1 + t_{0012})(1 + t_{0021}) \dots \\
 &= 1 + t_0 + t_1 - t_2 - (t_{01} - t_0t_1) + (t_{02} - t_0t_2) + (t_{12} - t_1t_2) \\
 &\quad + (t_{001} - t_0t_{01}) - (t_{002} - t_0t_{02}) + (t_{011} - t_1t_{01}) \\
 &\quad + (t_{022} - t_2t_{02}) - (t_{112} - t_1t_{12}) + (t_{122} - t_2t_{12}) \\
 &\quad - (t_{012} + t_{021} + t_0t_1t_2 - t_0t_{12} - t_1t_{02} - t_2t_{01}) \dots \quad \text{(H.61)}
 \end{aligned}$$

The form of the remaining cycle expansions depends crucially on the special role played by the boundary orbits: by (H.59) the orbit  $t_2$  does not contribute to  $A_2$  and

$B_1$ ,

$$\begin{aligned}
1/\zeta_{A_2} &= (1+t_0)(1-t_1)(1+t_{01})(1+t_{02})(1-t_{12}) \\
&\quad (1-t_{001})(1-t_{002})(1+t_{011})(1+t_{012})(1+t_{021})(1+t_{022})(1-t_{112}) \\
&\quad (1-t_{122})(1+t_{0001})(1+t_{0002})(1-t_{0011})(1-t_{0012})(1-t_{0021})\dots \\
&= 1+t_0-t_1+(t_{01}-t_0t_1)+t_{02}-t_{12} \\
&\quad -(t_{001}-t_0t_{01})-(t_{002}-t_0t_{02})+(t_{011}-t_1t_{01}) \\
&\quad +t_{022}-t_{122}-(t_{112}-t_1t_{12})+(t_{012}+t_{021}-t_0t_{12}-t_1t_{02})\dots \text{(H.62)}
\end{aligned}$$

and

$$\begin{aligned}
1/\zeta_{B_1} &= (1-t_0)(1+t_1)(1+t_{01})(1-t_{02})(1+t_{12}) \\
&\quad (1+t_{001})(1-t_{002})(1-t_{011})(1+t_{012})(1+t_{021})(1-t_{022})(1-t_{112}) \\
&\quad (1+t_{122})(1+t_{0001})(1-t_{0002})(1-t_{0011})(1+t_{0012})(1+t_{0021})\dots \\
&= 1-t_0+t_1+(t_{01}-t_0t_1)-t_{02}+t_{12} \\
&\quad +(t_{001}-t_0t_{01})-(t_{002}-t_0t_{02})-(t_{011}-t_1t_{01}) \\
&\quad -t_{022}+t_{122}-(t_{112}-t_1t_{12})+(t_{012}+t_{021}-t_0t_{12}-t_1t_{02})\dots \text{(H.63)}
\end{aligned}$$

In the above we have assumed that  $t_2$  does not change sign under  $C_{4v}$  reflections. For the mixed-symmetry subspace  $E$  the curvature expansion is given by

$$\begin{aligned}
1/\zeta_E &= 1+t_2+(-t_0^2+t_1^2)+(2t_{002}-t_2t_0^2-2t_{112}+t_2t_1^2) \\
&\quad +(2t_{0011}-2t_{0022}+2t_2t_{002}-t_{01}^2-t_{02}^2+2t_{1122}-2t_2t_{112} \\
&\quad +t_{12}^2-t_0^2t_1^2)+(2t_{00002}-2t_{00112}+2t_2t_{0011}-2t_{00121}-2t_{00211} \\
&\quad +2t_{00222}-2t_2t_{0022}+2t_{01012}+2t_{01021}-2t_{01102}-t_2t_{01}^2+2t_{02022} \\
&\quad -t_2t_{02}^2+2t_{11112}-2t_{11222}+2t_2t_{1122}-2t_{12122}+t_2t_{12}^2-t_2t_0^2t_1^2 \\
&\quad +2t_{002}(-t_0^2+t_1^2)-2t_{112}(-t_0^2+t_1^2)) \quad \text{(H.64)}
\end{aligned}$$

A quick test of the  $\zeta = \zeta_{A_1}\zeta_{A_2}\zeta_{B_1}\zeta_{B_2}\zeta_E^2$  factorization is afforded by the topological polynomial; substituting  $t_p = z^{n_p}$  into the expansion yields

$$1/\zeta_{A_1} = 1 - 3z, \quad 1/\zeta_{A_2} = 1/\zeta_{B_1} = 1, \quad 1/\zeta_{B_2} = 1/\zeta_E = 1 + z,$$

in agreement with (15.46).

exercise 20.9

## H.7 $C_{2v}$ factorization

An arrangement of four identical disks on the vertices of a rectangle has  $C_{2v}$  symmetry (figure H.2b).  $C_{2v}$  consists of  $\{e, \sigma_x, \sigma_y, C_2\}$ , i.e., the reflections across the symmetry axes and a rotation by  $\pi$ .

**Table H.2:**  $C_{2v}$  correspondence between the ternary  $\{0, 1, 2\}$  fundamental domain prime cycles  $\tilde{p}$  and the full 4-disk  $\{1,2,3,4\}$  cycles  $p$ , together with the  $C_{2v}$  transformation that maps the end point of the  $\tilde{p}$  cycle into an irreducible segment of the  $p$  cycle. The degeneracy of the  $p$  cycle is  $m_p = 4n_{\tilde{p}}/n_p$ . Note that the 012 and 021 cycles are related by time reversal, but cannot be mapped into each other by  $C_{2v}$  transformations. The full space orbit listed here is generated from the symmetry reduced code by the rules given in sect. H.7, starting from disk 1.

$\tilde{p}$	$p$	$\mathbf{g}$	$\tilde{p}$	$p$	$\mathbf{g}$
0	14	$\sigma_y$	0001	14143232	$C_2$
1	12	$\sigma_x$	0002	14142323	$\sigma_x$
2	13	$C_2$	0011	1412	$e$
01	1432	$C_2$	0012	14124143	$\sigma_y$
02	1423	$\sigma_x$	0021	14134142	$\sigma_y$
12	1243	$\sigma_y$	0022	1413	$e$
001	141232	$\sigma_x$	0102	14324123	$\sigma_y$
002	141323	$C_2$	0111	14343212	$C_2$
011	143412	$\sigma_y$	0112	14342343	$\sigma_x$
012	143	$e$	0121	14312342	$\sigma_x$
021	142	$e$	0122	14313213	$C_2$
022	142413	$\sigma_y$	0211	14212312	$\sigma_x$
112	121343	$C_2$	0212	14213243	$C_2$
122	124213	$\sigma_x$	0221	14243242	$C_2$
			0222	14242313	$\sigma_x$
			1112	12124343	$\sigma_y$
			1122	1213	$e$
			1222	12424313	$\sigma_y$

This system affords a rather easy visualization of the conversion of a 4-disk dynamics into a fundamental domain symbolic dynamics. An orbit leaving the fundamental domain through one of the axis may be folded back by a reflection on that axis; with these symmetry operations  $g_0 = \sigma_x$  and  $g_1 = \sigma_y$  we associate labels 1 and 0, respectively. Orbits going to the diagonally opposed disk cross the boundaries of the fundamental domain twice; the product of these two reflections is just  $C_2 = \sigma_x\sigma_y$ , to which we assign the label 2. For example, a ternary string 0010201... is converted into 12143123..., and the associated group-theory weight is given by ...  $g_1g_0g_2g_0g_1g_0g_0$ .

Short ternary cycles and the corresponding 4-disk cycles are listed in table H.2. Note that already at length three there is a pair of cycles (012 = 143 and 021 = 142) related by time reversal, but *not* by any  $C_{2v}$  symmetries.

The above is the complete description of the symbolic dynamics for 4 sufficiently separated equal disks placed at corners of a rectangle. However, if the fundamental domain requires further partitioning, the ternary description is insufficient. For example, in the stadium billiard fundamental domain one has to distinguish between bounces off the straight and the curved sections of the billiard wall; in that case five symbols suffice for constructing the covering symbolic dynamics.

The group  $C_{2v}$  has four 1-dimensional representations, distinguished by their behavior under axis reflections. The  $A_1$  representation is symmetric with respect to both reflections; the  $A_2$  representation is antisymmetric with respect to both. The  $B_1$  and  $B_2$  representations are symmetric under one and antisymmetric under



the other reflection. The character table is

$C_{2v}$	$A_1$	$A_2$	$B_1$	$B_2$
$e$	1	1	1	1
$C_2$	1	1	-1	-1
$\sigma_x$	1	-1	1	-1
$\sigma_y$	1	-1	-1	1

Substituted into the factorized determinant (21.14), the contributions of periodic orbits split as follows

$$\begin{array}{l}
 g_{\bar{p}} \\
 e: (1 - t_{\bar{p}})^4 = (1 - t_{\bar{p}}) (1 - t_{\bar{p}}) (1 - t_{\bar{p}}) (1 - t_{\bar{p}}) \\
 C_2: (1 - t_{\bar{p}}^2)^2 = (1 - t_{\bar{p}}) (1 - t_{\bar{p}}) (1 - t_{\bar{p}}) (1 - t_{\bar{p}}) \\
 \sigma_x: (1 - t_{\bar{p}}^2)^2 = (1 - t_{\bar{p}}) (1 + t_{\bar{p}}) (1 - t_{\bar{p}}) (1 + t_{\bar{p}}) \\
 \sigma_y: (1 - t_{\bar{p}}^2)^2 = (1 - t_{\bar{p}}) (1 + t_{\bar{p}}) (1 + t_{\bar{p}}) (1 - t_{\bar{p}})
 \end{array}$$

Cycle expansions follow by substituting cycles and their group theory factors from table H.2. For  $A_1$  all characters are +1, and the corresponding cycle expansion is given in (H.60). Similarly, the totally antisymmetric subspace factorization  $A_2$  is given by (H.61), the  $B_2$  factorization of  $C_{4v}$ . For  $B_1$  all  $t_p$  with an odd total number of 0's and 2's change sign:

$$\begin{aligned}
 1/\zeta_{B_1} &= (1 + t_0)(1 - t_1)(1 + t_2)(1 + t_{01})(1 - t_{02})(1 + t_{12}) \\
 &\quad (1 - t_{001})(1 + t_{002})(1 + t_{011})(1 - t_{012})(1 - t_{021})(1 + t_{022})(1 + t_{112}) \\
 &\quad (1 - t_{122})(1 + t_{0001})(1 - t_{0002})(1 - t_{0011})(1 + t_{0012})(1 + t_{0021}) \dots \\
 &= 1 + t_0 - t_1 + t_2 + (t_{01} - t_0 t_1) - (t_{02} - t_0 t_2) + (t_{12} - t_1 t_2) \\
 &\quad - (t_{001} - t_0 t_{01}) + (t_{002} - t_0 t_{02}) + (t_{011} - t_1 t_{01}) \\
 &\quad + (t_{022} - t_2 t_{02}) + (t_{112} - t_1 t_{12}) - (t_{122} - t_2 t_{12}) \\
 &\quad - (t_{012} + t_{021} + t_0 t_1 t_2 - t_0 t_{12} - t_1 t_{02} - t_2 t_{01}) \dots \quad (\text{H.65})
 \end{aligned}$$

For  $B_2$  all  $t_p$  with an odd total number of 1's and 2's change sign:

$$\begin{aligned}
 1/\zeta_{B_2} &= (1 - t_0)(1 + t_1)(1 + t_2)(1 + t_{01})(1 + t_{02})(1 - t_{12}) \\
 &\quad (1 + t_{001})(1 + t_{002})(1 - t_{011})(1 - t_{012})(1 - t_{021})(1 - t_{022})(1 + t_{112}) \\
 &\quad (1 + t_{122})(1 + t_{0001})(1 + t_{0002})(1 - t_{0011})(1 - t_{0012})(1 - t_{0021}) \dots \\
 &= 1 - t_0 + t_1 + t_2 + (t_{01} - t_0 t_1) + (t_{02} - t_0 t_2) - (t_{12} - t_1 t_2) \\
 &\quad + (t_{001} - t_0 t_{01}) + (t_{002} - t_0 t_{02}) - (t_{011} - t_1 t_{01}) \\
 &\quad - (t_{022} - t_2 t_{02}) + (t_{112} - t_1 t_{12}) + (t_{122} - t_2 t_{12}) \\
 &\quad - (t_{012} + t_{021} + t_0 t_1 t_2 - t_0 t_{12} - t_1 t_{02} - t_2 t_{01}) \dots \quad (\text{H.66})
 \end{aligned}$$

Note that all of the above cycle expansions group long orbits together with their pseudoorbit shadows, so that the shadowing arguments for convergence still apply.

The topological polynomial factorizes as

$$\frac{1}{\zeta_{A_1}} = 1 - 3z \quad , \quad \frac{1}{\zeta_{A_2}} = \frac{1}{\zeta_{B_1}} = \frac{1}{\zeta_{B_2}} = 1 + z,$$

consistent with the 4-disk factorization (15.46).

## H.8 Hénon map symmetries

We note here a few simple symmetries of the Hénon map (3.19). For  $b \neq 0$  the Hénon map is reversible: the backward iteration of (3.20) is given by

$$x_{n-1} = -\frac{1}{b}(1 - ax_n^2 - x_{n+1}). \quad (\text{H.67})$$

Hence the time reversal amounts to  $b \rightarrow 1/b$ ,  $a \rightarrow a/b^2$  symmetry in the parameter plane, together with  $x \rightarrow -x/b$  in the coordinate plane, and there is no need to explore the  $(a, b)$  parameter plane outside the strip  $b \in \{-1, 1\}$ . For  $b = -1$  the map is orientation and area preserving ,

$$x_{n-1} = 1 - ax_n^2 - x_{n+1}, \quad (\text{H.68})$$

the backward and the forward iteration are the same, and the non-wandering set is symmetric across the  $x_{n+1} = x_n$  diagonal. This is one of the simplest models of a Poincaré return map for a Hamiltonian flow. For the orientation reversing  $b = 1$  case we have

$$x_{n-1} = 1 - ax_n^2 + x_{n+1}, \quad (\text{H.69})$$

and the non-wandering set is symmetric across the  $x_{n+1} = -x_n$  diagonal.

## Commentary

**Remark H.1 Literature** This material is covered in any introduction to linear algebra [H.1, H.2, H.3] or group theory [21.15, 10.2]. The exposition given in sects. H.2.1 and H.2.2 is taken from refs. [4.9, 4.10, 9.4]. Who wrote this down first we do not know, but we like Harter's exposition [H.23, H.24, 4.15] best. Harter's theory of class algebras offers a more elegant and systematic way of constructing the maximal set of commuting invariant matrices  $\mathbf{M}_i$  than the sketch offered in this section.

**Remark H.2** Labeling conventions While there is a variety of labeling conventions [25.19, 9.24] for the reduced  $C_{4v}$  dynamics, we prefer the one introduced here because of its close relation to the group-theoretic structure of the dynamics: the global 4-disk trajectory can be generated by mapping the fundamental domain trajectories onto the full 4-disk space by the accumulated product of the  $C_{4v}$  group elements.

**Remark H.3**  $C_{2v}$  symmetry  $C_{2v}$  is the symmetry of several systems studied in the literature, such as the stadium billiard [8.10], and the 2-dimensional anisotropic Kepler potential [21.3].

## Exercises

H.1. **Am I a group?** Show that multiplication table

	$e$	$a$	$b$	$c$	$d$	$f$
$e$	$e$	$a$	$b$	$c$	$d$	$f$
$a$	$a$	$e$	$d$	$b$	$f$	$c$
$b$	$b$	$d$	$e$	$f$	$c$	$a$
$c$	$c$	$b$	$f$	$e$	$a$	$d$
$d$	$d$	$f$	$c$	$a$	$e$	$b$
$f$	$f$	$c$	$a$	$d$	$b$	$e$

describes a group. Or does it? (Hint: check whether this table satisfies the group axioms of appendix H.1.)

From W.G. Harter [4.15]

H.2. **Three coupled pendulums with a  $C_2$  symmetry.**

Consider 3 pendulums in a row: the 2 outer ones of the same mass  $m$  and length  $l$ , the one midway of same length but different mass  $M$ , with the tip coupled to the tips of the outer ones with springs of stiffness  $k$ . Assume displacements are small,  $x_i/l \ll 1$ .

(a) Show that the acceleration matrix  $\ddot{\mathbf{x}} = -\mathbf{a} \mathbf{x}$  is

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} = - \begin{bmatrix} a+b & -a & 0 \\ -c & 2c+b & -c \\ 0 & -a & a+b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

where  $a = k/ml$ ,  $c = k/Ml$  and  $b = g/l$ .

(b) Check that  $[\mathbf{a}, \mathbf{R}] = 0$ , i.e., that the dynamics is invariant under  $C_2 = \{e, R\}$ , where  $\mathbf{R}$  interchanges the outer pendulums,

$$\mathbf{R} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

(c) Construct the corresponding projection operators  $\mathbf{P}_+$  and  $\mathbf{P}_-$ , and show that the 3-pendulum system decomposes into a 1-dimensional subspace, with eigenvalue  $(\omega^{(-)})^2 = a + b$ , and a 2-dimensional subspace, with acceleration matrix (trust your own algebra, if it strays from what is stated here)

$$\mathbf{a}^{(+)} = \begin{bmatrix} a+b & -\sqrt{2}a \\ -\sqrt{2}c & c+b \end{bmatrix}.$$

The exercise is simple enough that you can do it without using the symmetry, so: construct  $\mathbf{P}_+$ ,  $\mathbf{P}_-$  first, use them to reduce  $\mathbf{a}$  to irreps, then proceed with computing remaining eigenvalues of  $\mathbf{a}$ .

(d) Does anything interesting happen if  $M = m$ ?

The point of the above exercise is that almost always the symmetry reduction is only partial: a matrix representation of dimension  $d$  gets reduced to a set of subspaces whose dimensions  $d^{(\alpha)}$  satisfy  $\sum d^{(\alpha)} = d$ . Beyond that, love many, trust few, and paddle your own canoe.

From W.G. Harter [4.15]

H.3. **Lorenz system in polar coordinates: dynamics.** (continuation of exercise 9.8.)

1. Show that (9.25) has two equilibria:

$$\begin{aligned} (r_0, z_0) &= (0, 0), & \theta_0 &\text{undefined} \\ (r_1, \theta_1, z_1) &= (\sqrt{2b(\rho-1)}, \pi/4, \rho) \quad (\text{H170}) \end{aligned}$$

2. Verify numerically that the eigenvalues and eigenvectors of the two equilibria are (we list here the precise numbers to help you check your programs):

$EQ_1 = (0, 12, 27)$  **equilibrium:** (and its  $C^{1/2}$ -rotation  $EQ_2$ ) has one stable real eigenvalue  $\lambda^{(1)} = -13.854578$ , and the unstable complex conjugate pair  $\lambda^{(2,3)} = \mu^{(2)} \pm i\omega^{(2)} = 0.093956 \pm i10.194505$ . The unstable eigenplane is defined by eigenvectors

$$\text{Re } \mathbf{e}^{(2)} = (-0.4955, -0.2010, -0.8450)$$

$$\text{Im } \mathbf{e}^{(2)} = (0.5325, -0.8464, 0)$$

with period  $T = 2\pi/\omega^{(2)} = 0.6163306$ ,

radial expansion multiplier

$$\Lambda_r = \exp(2\pi\mu^{(2)}/\omega^{(2)}) = 1.059617,$$

and the contracting multiplier

$$\Lambda_c = \exp(2\pi\mu^{(1)}/\omega^{(2)}) \approx 1.95686 \times 10^{-4}$$

along the stable eigenvector of  $EQ_1$ ,

$$\mathbf{e}^{(3)} = (0.8557, -0.3298, -0.3988).$$

$EQ_0 = (0, 0, 0)$  **equilibrium:** The stable eigenvector  $\mathbf{e}^{(1)} = (0, 0, 1)$  of  $EQ_0$ , has contraction rate  $\lambda^{(2)} = -b = -2.666\dots$

The other stable eigenvector is

$$\mathbf{e}^{(2)} = (-0.244001, -0.969775, 0), \text{ with}$$

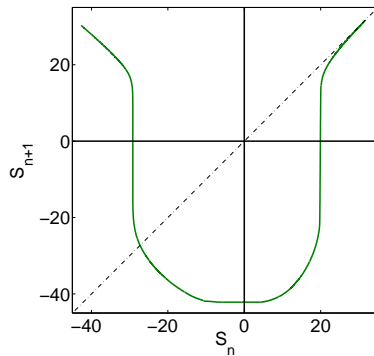
contracting eigenvalue

$$\lambda^{(2)} = -22.8277. \text{ The unstable eigenvector}$$

$$\mathbf{e}^{(3)} = (-0.653049, 0.757316, 0) \text{ has eigenvalue}$$

$$\lambda^{(3)} = 11.8277.$$

- Plot the Lorenz strange attractor both in the Lorenz coordinates figure 2.5, and in the doubled-polar angle coordinates (9.21) for the Lorenz parameter values  $\sigma = 10$ ,  $b = 8/3$ ,  $\rho = 28$ . Topologically, does it resemble the Lorenz butterfly, the Rössler attractor, or neither? The Poincaré section of the Lorenz flow fixed by the  $z$ -axis and the equilibrium in the doubled polar angle representation, and the corresponding Poincaré return map  $(s_n, s_{n+1})$  are plotted in figure 11.8.
- Construct the Poincaré return map  $(s_n, s_{n+1})$ ,



where  $s$  is arc-length measured along the unstable manifold of  $EQ_0$ , lower Poincaré section of figure 11.8(b). Elucidate its relation to the Poincaré return map of figure 11.9. (plot by J. Halcrow)

- Show that if a periodic orbit of the polar representation Lorenz is also periodic orbit of the Lorenz flow, their Floquet multipliers are the same. How do the Floquet multipliers of relative periodic orbits of the representations relate to each other?
- What does the volume contraction formula (4.48) look like now? Interpret.

#### H.4. Laplacian is a non-local operator.

While the Laplacian is a simple tri-diagonal difference operator (H.37), its inverse (the “free” propagator of statistical mechanics and quantum field theory) is a messier object. A way to compute is to start expanding propagator as a power series in the Laplacian

$$\frac{1}{m^2 \mathbf{1} - \Delta} = \frac{1}{m^2} \sum_{n=0}^{\infty} \frac{1}{m^{2n}} \Delta^n. \quad (\text{H.71})$$

As  $\Delta$  is a finite matrix, the expansion is convergent for sufficiently large  $m^2$ . To get a feeling for what is involved in evaluating such series, show that  $\Delta^2$  is:

$$\Delta^2 = \frac{1}{a^4} \begin{pmatrix} 6 & -4 & 1 & & & 1 & -4 \\ -4 & 6 & -4 & 1 & & & & \\ 1 & -4 & 6 & -4 & 1 & & & \\ & & 1 & -4 & \ddots & & & \\ & & & & & & 6 & -4 \\ -4 & 1 & & & & 1 & -4 & 6 \end{pmatrix}. \quad (\text{H.72})$$

What  $\Delta^3$ ,  $\Delta^4$ ,  $\dots$  contributions look like is now clear; as we include higher and higher powers of the Laplacian, the propagator matrix fills up; while the *inverse* propagator is differential operator connecting only the nearest neighbors, the propagator is integral operator, connecting every lattice site to any other lattice site.

This matrix can be evaluated as is, on the lattice, and sometime it is evaluated this way, but in case at hand a wonderful simplification follows from the observation that the lattice action is translationally invariant, exercise H.5.

- Lattice Laplacian diagonalized.** Insert the identity  $\sum \mathbf{P}^{(k)} = \mathbf{1}$  wherever you profitably can, and use the eigenvalue equation (H.49) to convert shift  $\mathbf{h}$  matrices into scalars. If  $\mathbf{M}$  commutes with  $\mathbf{h}$ , then  $(\varphi_k^\dagger \cdot \mathbf{M} \cdot \varphi_{k'}) = \tilde{M}^{(k)} \delta_{kk'}$ , and the matrix  $\mathbf{M}$  acts as a multiplication by the scalar  $\tilde{M}^{(k)}$  on the  $k$ th subspace. Show that for the 1-dimensional version of the lattice Laplacian (H.37) the projection on the  $k$ th subspace is

$$(\varphi_k^\dagger \cdot \Delta \cdot \varphi_{k'}) = \frac{2}{a^2} \left( \cos\left(\frac{2\pi}{N}k\right) - 1 \right) \delta_{kk'}. \quad (\text{H.73})$$

In the  $k$ th subspace the propagator is simply a number, and, in contrast to the mess generated by (H.71), there is nothing to evaluating:

$$\varphi_k^\dagger \frac{1}{m^2 \mathbf{1} - \Delta} \varphi_{k'} = \frac{\delta_{kk'}}{m^2 - \frac{2}{(ma)^2} (\cos 2\pi k/N - 1)}, \quad (\text{H.74})$$

where  $k$  is a site in the  $N$ -dimensional dual lattice, and  $a = L/N$  is the lattice spacing.

H.6. **Fix Predrag's lecture of Feb 5, 2008.** Are the  $C_3$  frequencies on pp. 4,5 correct? If not, write the correct expression for the beat frequency.

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